We derive a program which computes ${}^3log.N$, rounded down, for positive integer N. Our specification is

|[**con** N : int $\{N > 0\}$;
  var x : int;
  S
  $\{x \leq {}^3log.N \quad \wedge \quad {}^3log.N \; < \; x+1\}$
]|

Of the ${}^3log$ function we have the following

(0) $x = {}^3log.N \quad \equiv \quad 3^x = N$

We can thus rewrite our specification as

|[**con** N : int $\{N > 0\}$;
  var x : int;
  S
  $\{3^x \leq N \quad \wedge \quad N \; < \; 3^{x+1}\}$
]|

We can choose an invariant by deleting a conjunct. The negation of the deleted conjunct is then the guard of our repetition. We choose as invariant $P : 3^x \leq N$, which is established by $x := 0$. Negation of $N \; < \; 3^{x+1}$ yields as guard $B : 3^{x+1} \leq N$.

A bound function t should satisfy $P \wedge B \;\Rightarrow\; 0 \leq t$. Since P implies $0 \;\leq\; N - 3^x$, $N - 3^x$ looks like a good bound function. We observe

  $P.(x \;:=\; x+1)$
$=$ { substitution }
  $0 \leq x+1 \quad \wedge \quad 3^{x+1} \leq N$
$=$ { definition of B }
  $0 \leq x+1 \quad \wedge \quad B$
$\Leftarrow$ { widening }
  $0 \leq x \quad \wedge \quad B$

The last step is not implied by the invariant so we strengthen it as follows

$$P : \quad 0 \leq x \;\wedge\; 3^x \leq N$$

hence our solution

$\{0 \leq N\}$
$x := 0$
{invariant $P : \quad 0 \leq x \quad \wedge \quad 3^x \leq N$, bound: $N - 3^x$}
;**do** $3^{x+1} \leq N \quad \rightarrow \quad x \;:=\; x+1$ **od**
$\{3^x \leq N \quad \wedge \quad N \; < \; 3^{x+1}\}$

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