The maximum element

Problem

Derive a solution to the following :

$|[$ **con** N : int $\{0 \leq N\}$; f : **array**$[0..N)$ **of** int;

  var r : int;

    S

  $\{(0): \quad r = \langle \uparrow i :\ 0 \leq i < N :\ f.i \rangle\}$

$]|$,

where $\uparrow$ denotes the maximum.

Solution

The only way we have to establish the postcondition is the repetitive construct. For the invariant of the repetition we generalize (0) by introducing a fresh variable n, thus obtaining the conjunction of

(1) $r = \langle \uparrow i :\ 0 \leq i < n :\ f.i \rangle$,

(2) $0 \leq n \leq N$,

which we establish with $n, r := 0, -\infty$.

As is customary given our chosen technique we are heading for an S of the form

$|[$ **var** n : int;

  $n, r := 0, -\infty$ $\{(1) \wedge (2)\}$

  ; **do** $n \neq N \rightarrow$

    "Maintain (1) under $n := n+1$"

    $n := n+1$

   od $\{(1) \wedge n = N$ hence$\}$

$\{(0)\}$

$]|$.

As both the proofs of termination and maintenance of (2) under $n := n+1$ are standard and trivial we omit them.

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We refine "Maintain (1) under $n := n+1$" by observing

$|[$ (1) $\wedge$ (2) $\wedge$ $n \neq N$

$\triangleright$

  $\langle \uparrow i :\ 0 \leq i < n+1 :\ f.i \rangle$

$=$ { split off $i = n$, $n < N$; (1) }

$r \uparrow f.n$

$]|$.

As our program scheme ensures that $0 \leq n < N$ holds within the body of the loop $r := r \uparrow f.n$ is a suitable refinement for "Maintain (1) under $n := n+1$". Hence the final program

$|[$ **var** n : int;

$n, r := 0, -\infty$ $\{(1) \wedge (2)\}$

; **do** $n \neq N \rightarrow$

$r := r \uparrow f.n$

; $n := n + 1$

od $\{(1) \wedge n = N$ hence$\}$

$\{(0)\}$

$]|$.

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