

The maximum element

Problem

Derive a solution to the following :

```
[[ con  $N : \text{int } \{0 \leq N\}$ ;  $f : \text{array}[0..N)$  of int;  
   var  $r : \text{int}$ ;  
    $S$   
    $\{(0) : r = \langle \uparrow i : 0 \leq i < N : f.i \rangle\}$   
]],
```

where \uparrow denotes the maximum.

Solution

The only way we have to establish the postcondition is the repetitive construct. For the invariant of the repetition we generalize (0) by introducing a fresh variable n , thus obtaining the conjunction of

$$(1) \quad r = \langle \uparrow i : 0 \leq i < n : f.i \rangle \quad ,$$

$$(2) \quad 0 \leq n \leq N \quad ,$$

which we establish with $n, r := 0, -\infty$.

As is customary given our chosen technique we are heading for an S of the form

```
[[ var  $n : \text{int}$ ;  
    $n, r := 0, -\infty \{(1) \wedge (2)\}$   
   ; do  $n \neq N \rightarrow$   
       “Maintain (1) under  $n := n + 1$ ”  
        $n := n + 1$   
   od  $\{(1) \wedge n = N \text{ hence}\}$   
    $\{(0)\}$   
]].
```

As both the proofs of termination and maintenance of (2) under $n := n + 1$ are standard and trivial we omit them.

* * *

We refine “Maintain (1) under $n := n + 1$ ” by observing

```
[[ (1)  $\wedge$  (2)  $\wedge n \neq N$ 
```

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$$\begin{aligned} & \langle \uparrow i : 0 \leq i < n + 1 : f.i \rangle \\ = & \quad \{ \text{split off } i = n, n < N; (1) \} \end{aligned}$$

$r \uparrow f.n$

]].

As our program scheme ensures that $0 \leq n < N$ holds within the body of the loop $r := r \uparrow f.n$ is a suitable refinement for “Maintain (1) under $n := n + 1$ ”. Hence the final program

```
[[ var n : int;
   n, r := 0, -∞ {(1) ∧ (2)}
; do n ≠ N →
   r := r ↑ f.n
   ; n := n + 1
   od {(1) ∧ n = N hence}
{(0)}
]].
```

Birmingham, 19 October 2006

Eric E. Macaulay
eric@mathmeth.com