The maximum pair difference

Given integer array A of length N, $1 \leq N$, we are asked to compute the maximum pair difference for A. Formally, for natural numbers p and q and integer r our program establishes

$$R: \quad r \;=\; \langle \uparrow p,q : \; 0 \leq p < q < N : \; A.p - A.q \rangle$$

* * *

We investigate our standard invariant, P, which is the conjunction of

$$Pr: \quad r \;=\; \langle \uparrow p,q : \; 0 \leq p < q < n : \; A.p - A.q \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with $n,r \;:=\; 0,-\infty$. We are heading for a program of the form

$n,r, \;:=\; 0,-\infty$
; **do** $n \neq N \;\rightarrow$
"Increase n by 1 under invariance of P"
od

The proofs of termination and maintenance of Pn under $n := n+1$ are standard and trivial and we thus omit them.

* * *

We observe

|[*Context:* $P \;\wedge\; n \neq N$

$\langle \uparrow p,q : \; 0 \leq p < q < n+1 : \; A.p - A.q \rangle$

$=$ { split off $q = n$ requires $n < N$ which follows from $Pn \;\wedge\; n \neq N$ }

$\langle \uparrow p,q : \; 0 \leq p < q < n : \; A.p - A.q \rangle \;\uparrow\; \langle \uparrow p : \; 0 \leq p < n : \; A.p - A.n \rangle$

$=$ { Pr }

$r \;\uparrow\; \langle \uparrow p : \; 0 \leq p < n : \; A.p - A.n \rangle$

$=$ { $+$ over $\uparrow$ }

$r \;\uparrow\; (\langle \uparrow p : \; 0 \leq p < n : \; A.p \rangle - A.n)$

$=$ { Ps }

$r \;\uparrow\; (s - A.n)$

]|

where we strengthen P with

$Ps: \quad s = \langle \uparrow p: \quad 0 \leq p < n: \quad A.p \rangle$

We initially establish Ps with $n, s := 0, -\infty$.

At this point we have

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \{ \quad Pr \quad \wedge \quad Ps \quad \} \\
\qquad r \quad := \quad r \uparrow (s - A.n) \\
\qquad \{ \quad (n := n+1).Pr \quad \} \\
]|
\end{array}
$$

giving us a program of the form

$$
\begin{array}{l}
\quad n, r, s \quad := \quad 0, -\infty, -\infty \\
; \textbf{do} \quad n \neq N \quad \rightarrow \\
\qquad \text{"Maintain} \quad Ps \quad \text{under} \quad n, r \quad := \quad n+1, \quad r \uparrow (s - A.n) \text{ "} \\
\quad \textbf{od}
\end{array}
$$

* * *

We turn our attention to the maintenance of Ps . We observe

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \langle \uparrow p: \quad 0 \leq p < n+1: \quad A.p \rangle \\
\quad = \qquad \{ \quad \text{split off} \quad p = n \quad \text{, requires} \quad 0 \leq n \quad \text{which follows from} \quad Pn \quad \} \\
\qquad \langle \uparrow p: \quad 0 \leq p < n: \quad A.p \rangle \quad \uparrow \quad A.n \\
\quad = \qquad \{ \; Ps \; \} \\
\qquad s \uparrow A.n \\
]|
\end{array}
$$

We now have

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \{ \quad Ps \quad \} \quad s \quad := \quad s \uparrow A.n \quad \{ \quad (n := n+1).Ps \quad \} \\
]|
\end{array}
$$

hence our solution

$$
\begin{array}{lll}
n, r, s & := & 0, -\infty, -\infty \\
; \textbf{do} \quad n \neq N & \rightarrow & \\
\quad n, r, s & := & n+1, \quad r \uparrow (s - A.n), \quad s \uparrow A.n \\
\textbf{od} & &
\end{array}
$$

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