The maximum non-empty segment sum

Given integer array A of length N , $1 \leq N$, we are asked to compute the maximum non-empty segment sum for A . Formally, for natural numbers p and q and integer r our program establishes

$$R: \quad r = \langle \uparrow p, q : \; 0 \leq p < q \leq N : \; S.p.q \rangle$$

where for natural i , S is given by

$$S.p.q = \langle \Sigma\, i : \; p \leq i < q : \; A.i \rangle$$

* * *

We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r = \langle \uparrow p, q : \; 0 \leq p < q \leq n : \; S.p.q \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with $n, r \; := \; 0, -\infty$. We are heading for a program of the form

$n, r, \; := \; 0, -\infty$
; **do** $n \neq N \;\rightarrow$
"Increase n by 1 under invariance of P "
od

The proofs of termination and maintenance of Pn under $n := n+1$ are standard and trivial and we thus omit them.

* * *

We observe

|[*Context:* $P \;\wedge\; n \neq N$

$\langle \uparrow p, q : \; 0 \leq p < q \leq n+1 : \; S.p.q \rangle$

$=$ { split off $q = n+1$ requires $n+1 \leq N$ which follows from $Pn \;\wedge\; n \neq N$ }

$\langle \uparrow p, q : \; 0 \leq p < q \leq n : \; S.p.q \rangle \;\uparrow\; \langle \uparrow p : \; 0 \leq p < n+1 : \; S.p.(n+1) \rangle$

$=$ { Pr ; $(n := n+1).Ps$ }

$r \uparrow s$

]|

where we strengthen P with

$Ps: \quad s = \langle \uparrow p: \quad 0 \le p < n: \quad S.p.n \rangle$

We initially establish Ps with $n, s \quad := \quad 0, -\infty$.

At this point we have

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \{ \quad Pr \quad \wedge \quad (n \ := \ n+1).Ps \quad \} \\
\qquad r \quad := \quad r \uparrow s \\
\qquad \{ \quad (n \ := \ n+1).Pr \quad \} \\
]|
\end{array}
$$

giving us a program of the form

$$
\begin{array}{l}
\quad n, r, s \quad := \quad 0, -\infty, -\infty \\
; \mathbf{do} \quad n \neq N \quad \rightarrow \\
\qquad \text{"Establish} \quad (n \ := \ n+1).Ps \ \text{"} \\
\qquad ; n, r \quad := \quad n+1, \ r \uparrow s \\
\quad \mathbf{od}
\end{array}
$$

* * *

We turn our attention to the maintenance of Ps . We observe

$$
\begin{array}{ll}
|[& \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
& \langle \uparrow p: \quad 0 \le p < n+1: \quad S.p.(n+1) \rangle \\
= & \quad \{ \ \text{split off} \quad p = n \ , \text{requires} \quad 0 \le n \quad \text{which follows from} \quad Pn \ \} \\
& \langle \uparrow p: \quad 0 \le p < n: \quad S.p.(n+1) \rangle \quad \uparrow \quad S.n.(n+1) \\
= & \quad \{ \ \text{properties of} \quad S \ \} \\
& \langle \uparrow p: \quad 0 \le p < n: \quad S.p.n \quad + \quad A.n \rangle \quad \uparrow \quad A.n \\
= & \quad \{ \ + \ \text{over} \quad \uparrow \ \} \\
& (\langle \uparrow p: \quad 0 \le p < n: \quad S.p.n \rangle \quad + \quad A.n) \quad \uparrow \quad A.n \\
= & \quad \{ \ Ps \ \} \\
& (s \ + \ A.n) \quad \uparrow \quad A.n \\
]|
\end{array}
$$

We now have

$|[$ *Context:* $P \quad \wedge \quad n \neq N$

$$\{ \quad Ps \quad \} \quad s \ := \ (s \ + \ A.n) \ \uparrow \ A.n \quad \{ \quad (n \ := \ n+1).Ps \quad \}$$

$]|$

hence our solution

$n, r, s \quad := \quad 0, -\infty, -\infty$

; **do** $\quad n \neq N \quad \rightarrow$

$\qquad s \quad := \quad (s \ + \ A.n) \quad \uparrow \quad A.n$

$\qquad ; n, r \quad := \quad n+1, \ r \uparrow s$

od

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