

The maximum non-empty segment sum

Given integer array A of length N , $1 \leq N$, we are asked to compute the maximum non-empty segment sum for A . Formally, for natural numbers p and q and integer r our program establishes

$$R: \quad r = \langle \uparrow p, q: 0 \leq p < q \leq N: S.p.q \rangle$$

where for natural i , S is given by

$$S.p.q = \langle \Sigma i: p \leq i < q: A.i \rangle$$

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We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r = \langle \uparrow p, q: 0 \leq p < q \leq n: S.p.q \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with $n, r := 0, -\infty$. We are heading for a program of the form

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n, r, := 0, -∞
; do n ≠ N →
    “Increase n by 1 under invariance of P ”
od

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The proofs of termination and maintenance of Pn under $n := n + 1$ are standard and trivial and we thus omit them.

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We observe

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[[ Context: P ∧ n ≠ N
   ⟨↑ p, q: 0 ≤ p < q ≤ n + 1: S.p.q⟩
=   { split off q = n + 1 requires n + 1 ≤ N which follows from Pn ∧ n ≠ N }
   ⟨↑ p, q: 0 ≤ p < q ≤ n: S.p.q⟩ ↑ ⟨↑ p: 0 ≤ p < n + 1: S.p.(n + 1)⟩
=   { Pr ; (n := n + 1).Ps }
   r ↑ s
]]

```

where we strengthen P with

$$Ps : s = \langle \uparrow p : 0 \leq p < n : S.p.n \rangle$$

We initially establish Ps with $n, s := 0, -\infty$.

At this point we have

$$\begin{aligned} &[[\text{Context: } P \wedge n \neq N \\ &\quad \{ Pr \wedge (n := n + 1).Ps \} \\ &\quad r := r \uparrow s \\ &\quad \{ (n := n + 1).Pr \} \\ &]] \end{aligned}$$

giving us a program of the form

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n, r, s := 0, -∞, -∞
; do n ≠ N →
    "Establish (n := n + 1).Ps"
    ; n, r := n + 1, r ↑ s
od

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We turn our attention to the maintenance of Ps . We observe

$$\begin{aligned} &[[\text{Context: } P \wedge n \neq N \\ &\quad \langle \uparrow p : 0 \leq p < n + 1 : S.p.(n + 1) \rangle \\ &= \{ \text{split off } p = n, \text{ requires } 0 \leq n \text{ which follows from } Pn \} \\ &\quad \langle \uparrow p : 0 \leq p < n : S.p.(n + 1) \rangle \uparrow S.n.(n + 1) \\ &= \{ \text{properties of } S \} \\ &\quad \langle \uparrow p : 0 \leq p < n : S.p.n + A.n \rangle \uparrow A.n \\ &= \{ + \text{ over } \uparrow \} \\ &\quad (\langle \uparrow p : 0 \leq p < n : S.p.n \rangle + A.n) \uparrow A.n \\ &= \{ Ps \} \\ &\quad (s + A.n) \uparrow A.n \\ &]] \end{aligned}$$

We now have

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[[ Context: P ∧ n ≠ N
   { Ps } s := (s + A.n) ↑ A.n { (n := n + 1).Ps }
]]

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hence our solution

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n, r, s := 0, -∞, -∞
; do n ≠ N →
  s := (s + A.n) ↑ A.n
  ; n, r := n + 1, r ↑ s
od

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