

## The maximum squared difference

Given integer array  $A$  of length  $N$ ,  $1 \leq N$ , we are asked to compute the maximum squared difference for  $A$ . Formally, for natural numbers  $p$  and  $q$  and integer  $r$  our program establishes

$$R: \quad r = \langle \uparrow p, q: 0 \leq p < q < N: (A.p - A.q)^2 \rangle$$

\* \* \*

Jeremy Weissmann conjectures that  $\langle \uparrow i: : i^2 \rangle = (\langle \uparrow i: : i \rangle)^2 \uparrow (\langle \uparrow i: : -i \rangle)^2$  and we therefore have to compute the maximum of  $r0^2$  and  $r1^2$  where  $r0$  is defined by

$$R0: \quad r0 = \langle \uparrow p, q: 0 \leq p < q < N: A.p - A.q \rangle$$

and  $r1$  is

$$R1: \quad r1 = \langle \uparrow p, q: 0 \leq p < q < N: A.q - A.p \rangle$$

We focus on computing  $r0$  first.

\* \* \*

We investigate our standard invariant,  $P$ , which is the conjunction of

$$Pr0: \quad r0 = \langle \uparrow p, q: 0 \leq p < q < n: A.p - A.q \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with  $n, r0 := 0, -\infty$ . We are heading for a program of the form

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n, r0, := 0, -∞
; do n ≠ N →
    “Increase n by 1 under invariance of P ”
od

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The proofs of termination and maintenance of  $Pn$  under  $n := n + 1$  are standard and trivial and we thus omit them.

\* \* \*

We observe

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[[ Context: P ∧ n ≠ N
   ⟨↑ p, q: 0 ≤ p < q < n + 1: A.p - A.q⟩
=   { split off q = n requires n < N which follows from Pn ∧ n ≠ N }
   ⟨↑ p, q: 0 ≤ p < q < n: A.p - A.q⟩ ↑ ⟨↑ p: 0 ≤ p < n: A.p - A.n⟩
=   { Pr0 ; + over ↑ ; Ps0 }
   r0 ↑ (s0 - A.n)
]]

```

where we strengthen  $P$  with

$$Ps0 : \quad s0 = \langle \uparrow p : 0 \leq p < n : A.p \rangle$$

We establish  $Ps0$  initially with  $n, s0 := 0, -\infty$ .

At this point we have

$$\begin{aligned} &[[ \textit{Context}: P \wedge n \neq N \\ &\quad \{ Pr0 \wedge Ps0 \} \\ &\quad r0 := r0 \uparrow (s0 - A.n) \\ &\quad \{ (n := n + 1).Pr0 \} \\ &]] \end{aligned}$$

giving us a program of the form

$$\begin{aligned} &n, r0, s0 := 0, -\infty, -\infty \\ &; \mathbf{do} \quad n \neq N \rightarrow \\ &\quad \text{“Maintain } Ps0 \text{ under } n, r0 := n + 1, r0 \uparrow (s0 - A.n) \text{”} \\ &\mathbf{od} \end{aligned}$$

\* \* \*

We turn our attention to the maintenance of  $Ps0$ . We observe

$$\begin{aligned} &[[ \textit{Context}: P \wedge n \neq N \\ &\quad \langle \uparrow p : 0 \leq p < n + 1 : A.p \rangle \\ &= \{ \text{split off } p = n, \text{ requires } 0 \leq n \text{ which follows from } Pn \} \\ &\quad \langle \uparrow p : 0 \leq p < n : A.p \rangle \uparrow A.n \\ &= \{ Ps0 \} \\ &\quad s \uparrow A.n \\ &]] \end{aligned}$$

We now have

$$\begin{aligned} &[[ \textit{Context}: P \wedge n \neq N \\ &\quad \{ Ps0 \} \quad s := s \uparrow A.n \quad \{ (n := n + 1).Ps0 \} \\ &]] \end{aligned}$$

giving us

```

n, r0, s0 := 0, -∞, -∞
; do n ≠ N →
    ; n, r0, s0 := n + 1, r0 ↑ (s0 - A.n), s ↑ A.n
od

```

The computation of  $Pr1$  is similar. Strengthening  $P$  with invariants

$$Pr1: \quad r1 = \langle \uparrow p, q: 0 \leq p < q < n: A.q - A.p \rangle$$

$$Ps1: \quad s1 = \langle \uparrow p: 0 \leq p < n: -A.p \rangle$$

a solution is

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n := 0
; r0, s0 := -∞, -∞
; r1, s1 := -∞, -∞
; do n ≠ N →
    ; r0, s0 := r0 ↑ (s0 - A.n), s ↑ A.n
    ; r1, s1 := r1 ↑ (s0 + A.n), s ↑ -A.n
    ; n := n + 1
od
; r := r02 ↑ r12

```

*E. Emmanuel Macaulay*  
eric@mathmeth.com

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