

The existence of the true/false pivot

Given boolean array A of length N , $0 \leq N$, we are asked to compute the existence of the true/false pivot for A . Formally, for natural number p and boolean r our program establishes

$$R: \quad r \equiv \langle \exists p: 0 \leq p \leq N: X.0.p \wedge Y.p.N \rangle$$

where, for natural i, q , X is defined as

$$X.p.q = \langle \forall i: p \leq i < q: A.i \rangle$$

and Y is defined as

$$Y.p.q = \langle \forall i: p \leq i < q: \neg A.i \rangle$$

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We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r \equiv \langle \exists p: 0 \leq p \leq N: X.0.p \wedge Y.p.n \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with $n, r := 0, true$. We are heading for a program of the form

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n, r, := 0, true
; do n ≠ N →
    "Increase n by 1 under invariance of P"
od

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The proofs of termination and maintenance of Pn under $n := n + 1$ are standard and trivial and we thus omit them.

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We observe

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[[ Context: P ∧ n ≠ N
   ⟨ ∃ p: 0 ≤ p ≤ n + 1: X.0.p ∧ Y.p.(n + 1) ⟩
   = { split off p = n + 1, requires 0 ≤ n + 1 which follows from Pn ∧ n ≠ N }
   ⟨ ∃ p: 0 ≤ p ≤ n: X.0.p ∧ Y.p.(n + 1) ⟩ ∨ (X.0.(n + 1) ∧ Y.(n + 1).(n + 1))
   = { properties of Y }
   ⟨ ∃ p: 0 ≤ p ≤ n: X.0.p ∧ Y.p.n ∧ ¬A.n ⟩ ∨ (X.0.(n + 1) ∧ true)
   = { predicate calculus; Pr; (n := n + 1).Ps }
   (r ∧ ¬A.n) ∨ s
]]

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where we strengthen P with

$$Ps : s = X.0.n$$

We establish Ps initially with $n, s := 0, true$.

At this point we have

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad \{ Pr \wedge (n := n + 1).Ps \} \\ &\quad r := (r \wedge \neg A.n) \vee s \\ &\quad \{ (n := n + 1).Pr \} \\ &]] \end{aligned}$$

giving us a program of the form

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n, r, s := 0, true, true
; do n ≠ N →
    “Establish (n := n + 1).Ps ”
    ; n, r := n + 1, (r ∧ ¬A.n) ∨ s
od

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We turn our attention to maintenance of Ps . We observe

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad X.0.(n + 1) \\ &= \{ \text{properties of } X \} \\ &\quad X.0.n \wedge A.n \\ &= \{ Ps \} \\ &\quad s \wedge A.n \\ &]] \end{aligned}$$

We now have

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad \{ Ps \} \quad s := s \wedge A.n \quad \{ (n := n + 1).Ps \} \\ &]] \end{aligned}$$

Hence our solution

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 $n, r, s := 0, true, true$   
; do  $n \neq N \rightarrow$   
     $s := s \wedge A.n$   
    ;  $n, r := n + 1, (r \wedge \neg A.n) \vee s$   
od
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E. Emmanuel Macaulay
eric@mathmeth.com

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