The existence of the true/false pivot

Given boolean array A of length N , $0 \leq N$, we are asked to compute the existence of the true/false pivot for A . Formally, for natural number p and boolean r our program establishes

$R: \quad r \quad \equiv \quad \langle \exists p: \quad 0 \ \leq \ p \ \leq \ N: \quad X.0.p \ \wedge \ Y.p.N \rangle$

where, for natural $i \quad q$, X is defined as

$X.p.q \quad = \quad \langle \forall i: \quad p \ \leq \ i \ < \ q: \quad A.i \rangle$

and Y is defined as

$Y.p.q \quad = \quad \langle \forall i: \quad p \ \leq \ i \ < \ q: \quad \neg A.i \rangle$

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We investigate our standard invariant, P , which is the conjunction of

$Pr: \quad r \quad \equiv \quad \langle \exists p: \quad 0 \ \leq \ p \ \leq \ N: \quad X.0.p \ \wedge \ Y.p.n \rangle$

$Pn: \quad 0 \ \leq \ n \ \leq \ N$

which we initially establish with $n, r \quad := \quad 0, \ true$. We are heading for a program of the form

$n, r, \quad := \quad 0, \ true$
; **do** $n \neq N \quad \rightarrow$
"Increase n by 1 under invariance of P "
od

The proofs of termination and maintenance of Pn under $n \ := \ n+1$ are standard and trivial and we thus omit them.

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We observe

|[*Context:* $P \quad \wedge \quad n \neq N$
$\langle \exists p: \quad 0 \ \leq \ p \ \leq \ n+1: \quad X.0.p \ \wedge \ Y.p.(n+1) \rangle$
$=$ { split off $p \ = \ n+1$, requires $0 \ \leq \ n+1$ which follows from $Pn \quad \wedge \quad n \neq N$ }
$\langle \exists p: \quad 0 \ \leq \ p \ \leq \ n: \quad X.0.p \ \wedge \ Y.p.(n+1) \rangle \quad \vee \quad (X.0.(n+1) \ \wedge \ Y.(n+1).(n+1))$
$=$ { properties of Y }
$\langle \exists p: \quad 0 \ \leq \ p \ \leq \ n: \quad X.0.p \ \wedge \ Y.p.n \ \wedge \ \neg A.n \rangle \quad \vee \quad (X.0.(n+1) \ \wedge \ true)$
$=$ { predicate calculus ; Pr ; $(n \ := \ n+1).Ps$ }
$(r \ \wedge \ \neg A.n) \ \vee \ s$
]|

where we strengthen P with

$Ps: \quad s \;=\; X.0.n$

We establish Ps initially with $n, s \;:=\; 0,\ true$.

At this point we have

$|[$ *Context:* $P \;\wedge\; n \neq N$

$\{\ Pr \;\wedge\; (n \;:=\; n+1).Ps\ \}$

$r \;:=\; (r \;\wedge\; \neg A.n) \;\vee\; s$

$\{\ (n \;:=\; n+1).Pr\ \}$

$]|$

giving us a program of the form

$n, r, s \;:=\; 0,\ true,\ true$

; **do** $n \neq N \;\rightarrow$

"Establish $(n \;:=\; n+1).Ps$ "

; $n, r \;:=\; n+1,\ (r \;\wedge\; \neg A.n) \;\vee\; s$

od

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We turn our attention to maintenance of Ps . We observe

$|[$ *Context:* $P \;\wedge\; n \neq N$

$X.0.(n+1)$

$=$ $\{$ properties of X $\}$

$X.0.n \;\wedge\; A.n$

$=$ $\{\ Ps\ \}$

$s \;\wedge\; A.n$

$]|$

We now have

$|[$ *Context:* $P \;\wedge\; n \neq N$

$\{\ Ps\ \}$ $s \;:=\; s \;\wedge\; A.n$ $\{\ (n \;:=\; n+1).Ps\ \}$

$]|$

Hence our solution

$$
\begin{array}{lll}
n, r, s & := & 0, \quad true, \quad true \\
; \textbf{do} \quad n \neq N & \rightarrow & \\
\quad s & := & s \ \wedge \ A.n \\
\quad ; n, r & := & n+1, \quad (r \ \wedge \ \neg A.n) \ \vee \ s \\
\textbf{od} & &
\end{array}
$$

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