Maximal credit

Given integer array A of length N , $0 \leq N$, we are asked to compute the maximal credit of A . Formally, for natural numbers p and q and integer r our program must establish

$$R: \quad r \;=\; \langle\uparrow p,q: \;\; 0 \;\leq\; p \;\leq\; q \;\leq\; N: \;\; C.p.q\rangle$$

where, for natural i , C is defined as

$$C.p.q \;=\; \langle\#i: \;\; p \;\leq\; i \;<\; q: \;\; A.i \;>\; 0\rangle - \langle\#i: \;\; p \;\leq\; i < q: \;\; A.i \;<\; 0\rangle$$

* * *

We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r \;=\; \langle\uparrow p,q: \;\; 0 \;\leq\; p \;\leq\; q \;\leq\; n: \;\; C.p.q\rangle$$

$$Pn: \quad 0 \;\leq\; n \;\leq\; N$$

which we initially establish with $n,r \;:=\; 0,0$. We are heading for a program of the form

$n,r, \;\; := \;\; 0,0$
; **do** $n \neq N \;\; \rightarrow$
  "Increase n by 1 under invariance of P "
 od

The proofs of termination and maintenance of Pn under $n \;:=\; n+1$ are standard and trivial and we thus omit them.

* * *

We observe

|[*Context:* $P \;\; \wedge \;\; n \neq N$
  $\langle\uparrow p,q: \;\; 0 \;\leq\; p \;\leq\; q \;\leq\; n+1: \;\; C.p.q\rangle$
 $=$ { split off $q \;=\; n+1$, requires $n+1 \;\leq\; N$ which follows from $Pn \;\; \wedge \;\; n \neq N$; Pr }
  $r \;\; \uparrow \;\; \langle\uparrow p: \;\; 0 \;\leq\; p \;\leq\; n+1: \;\; C.p.(n+1)\rangle$
 $=$ { assuming $(n \;:=\; n+1).Ps$ }
  $r \uparrow s$
]|

where we strengthen P with

$$Ps: \quad s \;=\; \langle\uparrow p: \;\; 0 \;\leq\; p \;\leq\; n: \;\; C.p.n\rangle$$

which we establish initially with $n, s \quad := \quad 0, 0$.

At this point we have

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \{ \quad Pr \quad \wedge \quad (n \;:=\; n+1).Ps \quad \} \qquad r \quad := \quad r \uparrow s \qquad \{ \quad (n \;:=\; n+1).Pr \quad \} \\
]|
\end{array}
$$

giving us a program of the form

$$
\begin{array}{l}
\quad n, r, \quad := \quad 0, 0 \\
; \,\mathbf{do} \quad n \neq N \quad \rightarrow \\
\qquad \text{“Establish} \quad (n \quad := \quad n+1).Ps \text{ ”} \\
\qquad ; n, r \quad := \quad n+1, \quad r \uparrow s \\
\quad \mathbf{od}
\end{array}
$$

* * *

We turn our attention to maintenance of Ps . We observe

$$
\begin{array}{l}
|[\quad \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n+1 : \quad C.p.(n+1) \rangle \\
= \qquad \{ \quad \text{split off} \quad p \;=\; n+1 \;\text{, requires} \quad 0 \leq n \quad \wedge \quad n+1 \;\leq\; N \quad \text{which follow from} \quad Pn \; \} \\
\qquad \langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad C.p.(n+1) \rangle \quad \uparrow \quad C.(n+1).(n+1) \\
= \qquad \{ \quad \text{properties of} \quad C \quad ; \quad + \quad \text{over} \quad \uparrow \} \\
\qquad \begin{cases}
(\langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad C.p.n \rangle \quad + \quad 1) \quad \uparrow \quad 0, & \text{if } 0 \;<\; A.n \\
(\langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad C.p.n \rangle \quad - \quad 1) \quad \uparrow \quad 0, & \text{if } A.n \;<\; 0 \\
(\langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad C.p.n \rangle) \quad \uparrow \quad 0, & \text{if } A.n \;=\; 0
\end{cases} \\
= \qquad \{ \; Ps \; \} \\
\qquad \begin{cases}
(s \;+\; 1) \quad \uparrow \quad 0, & \text{if } 0 \;<\; A.n \\
(s \;-\; 1) \quad \uparrow \quad 0, & \text{if } A.n \;<\; 0 \\
s \;\uparrow\; 0, & \text{if } A.n \;=\; 0
\end{cases} \\
]|
\end{array}
$$

We now have

$|[$ *Context:* $P \quad \wedge \quad n \neq N$

$\{ \quad Ps \quad \}$

$\quad$ **if** $0 < A.n \quad \rightarrow \quad s := (s + 1) \uparrow 0$

$\quad [] \quad A.n < 0 \quad \rightarrow \quad s := (s - 1) \uparrow 0$

$\quad [] \quad A.n = 0 \quad \rightarrow \quad s := s \uparrow 0$

$\quad$ **fi**

$\{ \quad (n := n + 1).Ps \quad \}$

$]|$

Hence our solution

$n, r, s \quad := \quad 0, 0, 0$

; **do** $n \neq N \quad \rightarrow$

$\quad$ **if** $0 < A.n \quad \rightarrow \quad s := (s + 1) \uparrow 0$

$\quad [] \quad A.n < 0 \quad \rightarrow \quad s := (s - 1) \uparrow 0$

$\quad [] \quad A.n = 0 \quad \rightarrow \quad s := s \uparrow 0$

$\quad$ **fi**

$\quad ; n, r \quad := \quad n + 1, \quad r \uparrow s$

od

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