The maximum segment product

Given integer array A of length N we are asked to compute the maximum segment product of A . More formally, for natural numbers p and q and integer r our program must establish

$$R: \quad r \;=\; \langle \uparrow p,q: \quad 0 \le p \le q \le N: \quad M.p.q \rangle$$

where, for natural i , M is defined as

$$M.p.q \;=\; \langle \Pi\, i:\; p \le i < q:\; A.i \rangle$$

*   *   *

We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r \;=\; \langle \uparrow p,q: \quad 0 \le p \le q \le n: \quad M.p.q \rangle$$

$$Pn: \quad 0 \le n \le N$$

which we initially establish with $n, r \;:=\; 0, 1$.

We progress to termination by incrementing n by one. The proofs of termination and maintenance of Pn under $n \;:=\; n+1$ are standard and trivial and we thus omit them.

*   *   *

We observe

$|[$ *Context:* $P \quad \wedge \quad n \neq N$

$\quad \langle \uparrow p,q: \quad 0 \;\le\; p \;\le\; q \;\le\; n+1: \quad M.p.q \rangle$

$=\quad \{$ split off $q \;=\; n+1$, requires $n+1 \;\le\; N$; Pr $\}$

$\quad r \;\uparrow\; \langle \uparrow p: \quad 0 \;\le\; p \;\le\; n+1: \quad M.p.(n+1) \rangle$

$=\quad \{$ assuming $(n \;:=\; n+1).Ps$ $\}$

$\quad r \uparrow s$

$]|$

We strengthen P with

$$Ps: \quad s \;=\; \langle \uparrow p: \quad 0 \;\le\; p \;\le\; n: \quad M.p.n \rangle$$

which we establish initially with $s := 1$.

At this point we have

$$
\begin{array}{l}
|[\; \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \{Pr \;\wedge\; (n \;:=\; n+1).Ps\} \quad r \;:=\; r \uparrow s \quad \{(n \;:=\; n+1).Pr\} \\
]|
\end{array}
$$

* * *

We turn our attention to maintenance of Ps . We observe

$$
\begin{array}{ll}
\multicolumn{2}{l}{|[\; \textit{Context:} \quad P \quad \wedge \quad n \neq N} \\
 & \langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n+1 : \quad M.p.(n+1) \rangle \\
= & \qquad \{ \text{ split off } \; p \;=\; n+1 \text{ , requires } \; n+1 \;\leq\; N \; \} \\
 & \langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad M.p.(n+1) \rangle \;\uparrow\; M.(n+1).(n+1) \\
= & \qquad \{ \text{ def of } \; M \; \} \\
 & \langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad M.p.n \;*\; A.n \rangle \;\uparrow\; 1 \\
= & \qquad \{ \text{ case analysis } \; \} \\
 & \begin{cases} (\langle \uparrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad M.p.n \rangle \;*\; A.n) \quad \uparrow \quad 1, & \text{if } 0 \;\leq\; A.n \\ (\langle \downarrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad M.p.n \rangle \;*\; A.n) \quad \uparrow \quad 1, & \text{if } A.n \;\leq\; 0 \end{cases} \\
= & \qquad \{ \; Ps \; ; \; Pt \; \} \\
 & \begin{cases} (s \;*\; A.n) \quad \uparrow \quad 1, & \text{if } 0 \leq A.n \\ (t \;*\; A.n) \quad \uparrow \quad 1, & \text{if } A.n \leq 0 \end{cases} \\
\multicolumn{2}{l}{]|}
\end{array}
$$

Once again we have strengthened P , this time with

$Pt: \qquad t \;=\; \langle \downarrow p : \quad 0 \;\leq\; p \;\leq\; n : \quad M.p.n \rangle$

We now have

$$
\begin{array}{l}
|[\; \textit{Context:} \quad P \;\wedge\; n \neq N \\
\quad \{Ps \;\wedge\; Pt\} \\
\qquad \textbf{if } \; 0 \;\leq\; A.n \quad \rightarrow \quad s \quad := \quad (s \;*\; A.n) \uparrow 1 \\
\qquad [\,] \;\; A.n \;\leq\; 0 \quad \rightarrow \quad s \;:=\; (t \;*\; A.n) \uparrow 1 \\
\qquad \textbf{fi} \\
\quad \{(n \;:=\; n+1).Ps\} \\
]|
\end{array}
$$

* * *

Let us now address the maintenance of Pt . We observe

$$
\begin{array}{l}
|[\ \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\quad \langle \downarrow p : \quad 0 \ \leq \ p \ \leq \ n+1 : \quad M.p.(n+1) \rangle \\
= \qquad \{ \text{ split off } \ p \ = \ n+1 \ \} \\
\quad \langle \downarrow p : \quad 0 \ \leq \ p \ \leq \ n : \quad M.p.(n+1) \rangle \quad \downarrow \quad M.(n+1).(n+1) \\
= \qquad \{ \text{ def of } M \ \} \\
\quad \langle \downarrow p : \quad 0 \ \leq \ p \ \leq \ n : \quad M.p.n \ * \ A.n)) \rangle \quad \downarrow \quad 1 \\
= \qquad \{ \text{ case analysis } \ \} \\
\quad \begin{cases} (\langle \downarrow p : \quad 0 \ \leq \ p \ \leq \ n : \quad M.p.n \rangle \ * \ A.n) \quad \downarrow \quad 1, & \text{if } 0 \ \leq \ A.n \\ (\langle \uparrow p : \quad 0 \ \leq \ p \ \leq \ n : \quad M.p.n \rangle \ * \ A.n) \quad \downarrow \quad 1, & \text{if } A.n \ \leq \ 0 \end{cases} \\
= \qquad \{ \ Ps \ ; \ \ Pt \ \} \\
\quad \begin{cases} (t \ * \ A.n) \quad \downarrow \quad 1, & \text{if } 0 \leq A.n \\ (s \ * \ A.n) \quad \downarrow \quad 1, & \text{if } A.n \leq 0 \end{cases} \\
]|
\end{array}
$$

This gives us

$$
\begin{array}{l}
|[\ \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\ \{ Ps \ \wedge \ Pt \} \\
\qquad \textbf{if } \ 0 \ \leq \ A.n \quad \rightarrow \quad t \quad := \quad (t \ * \ A.n) \downarrow 1 \\
\qquad [] \ \ A.n \ \leq \ 0 \quad \rightarrow \quad t \ := \ (s \ * \ A.n) \downarrow 1 \\
\qquad \textbf{fi} \\
\ \{ (n \ := \ n+1).Pt \} \\
]|
\end{array}
$$

Our solution is thus

$$
\begin{array}{l}
n, r, s, t \ := \ 0, 1, 1, 1 \\
; \textbf{do} \ \ n \neq N \quad \rightarrow \\
\qquad \textbf{if } \ 0 \leq A.n \quad \rightarrow \quad s, t \quad := \quad (s \ * \ A.n) \uparrow 1, \quad (t \ * \ A.n) \downarrow 1 \\
\qquad [] \ A.n \ \leq \ 0 \quad \rightarrow \quad s, t \quad := \quad (t \ * \ A.n) \uparrow 1, \quad (s \ * \ A.n) \downarrow 1 \\
\qquad \textbf{fi} \\
\qquad ; r, n \quad := \quad r \uparrow s, \quad n+1 \\
\quad \textbf{od}
\end{array}
$$

E. Emmanuel Macaulay
eric@mathmeth.com

16th January 2007