The maximum segment product

Given integer array A of length N we are asked to compute the maximum segment product of A . More formally, for natural numbers p and q and integer r our program must establish

$$R: \quad r \;=\; \langle \uparrow p,q : \quad 0 \le p \le q \le N : \quad M.p.q \rangle$$

where, for natural i , M is defined as

$$M.p.q \;=\; \langle \Pi\, i : \; p \le i < q : \; A.i \rangle$$

* * *

We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r \;=\; \langle \uparrow p,q : \quad 0 \le p \le q \le n : \quad M.p.q \rangle$$

$$Pn: \quad 0 \le n \le N$$

which we initially establish with $n,r \;:=\; 0,1$.

We progress to termination by incrementing n by one. The proofs of termination and maintenance of Pn under $n \;:=\; n+1$ are standard and trivial and we thus omit them.

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We observe

$| [$ *Context:* $P \quad \wedge \quad n \neq N$

$\quad \langle \uparrow p,q : \quad 0 \;\le\; p \;\le\; q \;\le\; n+1 : \quad M.p.q \rangle$

$= \qquad \{$ split off $q \;=\; n+1$, requires $n+1 \;\le\; N$; $Pr \}$

$\quad r \;\uparrow\; \langle \uparrow p : \quad 0 \;\le\; p \;\le\; n+1 : \quad M.p.(n+1) \rangle$

$= \qquad \{$ assuming $(n \;:=\; n+1).Ps \}$

$\quad r \uparrow s$

$] |$

We strengthen P with

$$Ps: \quad s \;=\; \langle \uparrow p : \quad 0 \;\le\; p \;\le\; n : \quad M.p.n \rangle$$

which we establish initially with $s := 1$.

At this point we have

$|[$ *Context:* $P \;\wedge\; n \neq N$

$\{Pr \;\wedge\; (n \;:=\; n+1).Ps\} \quad r \;:=\; r \uparrow s \quad \{(n \;:=\; n+1).Pr\}$

$]|$

* * *

We turn our attention to maintenance of Ps . We observe

$|[$ *Context:* $P \;\wedge\; n \neq N$

$\langle \uparrow p : \;\; 0 \;\leq\; p \;\leq\; n+1 : \;\; M.p.(n+1) \rangle$

$=$ { split off $p \;=\; n+1$, requires $n+1 \;\leq\; N$ }

$\langle \uparrow p : \;\; 0 \;\leq\; p \;\leq\; n : \;\; M.p.(n+1) \rangle \;\uparrow\; M.(n+1).(n+1)$

$=$ { def of M }

$\langle \uparrow p : \;\; 0 \;\leq\; p \;\leq\; n : \;\; M.p.n \;*\; A.n \rangle \;\uparrow\; 1$

$=$ { case analysis }

$$\begin{cases} (\langle \uparrow p : \;\; 0 \;\leq\; p \;\leq\; n : \;\; M.p.n \rangle \;*\; A.n) \;\;\uparrow\;\; 1, & \text{if } 0 \;\leq\; A.n \\ (\langle \downarrow p : \;\; 0 \;\leq\; p \;\leq\; n : \;\; M.p.n \rangle \;*\; A.n) \;\;\uparrow\;\; 1, & \text{if } A.n \;\leq\; 0 \end{cases}$$

$=$ { Ps ; Pt }

$$\begin{cases} (s \;*\; A.n) \;\;\uparrow\;\; 1, & \text{if } 0 \leq A.n \\ (t \;*\; A.n) \;\;\uparrow\;\; 1, & \text{if } A.n \leq 0 \end{cases}$$

$]|$

Once again we have strengthened P , this time with

$Pt: \qquad t \;=\; \langle \downarrow p : \;\; 0 \;\leq\; p \;\leq\; n : \;\; M.p.n \rangle$

We now have

$|[$ *Context:* $P \;\wedge\; n \neq N$

$\{Ps \;\wedge\; Pt\}$

if $0 \;\leq\; A.n \;\;\rightarrow\;\; s \;\;:=\;\; (s \;*\; A.n) \uparrow 1$

$[] \;\; A.n \;\leq\; 0 \;\;\rightarrow\;\; s \;:=\; (t \;*\; A.n) \uparrow 1$

fi

$\{(n \;:=\; n+1).Ps\}$

$]|$

* * *

Let us now address the maintenance of Pt . We observe

$$
\begin{array}{l}
|[\ \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\qquad \langle \downarrow p : \quad 0 \ \le\ p\ \le\ n+1 : \quad M.p.(n+1) \rangle \\
\quad = \qquad \{ \text{ split off } p \ =\ n+1 \ \} \\
\qquad \langle \downarrow p : \quad 0 \ \le\ p\ \le\ n : \quad M.p.(n+1) \rangle \quad \downarrow \quad M.(n+1).(n+1) \\
\quad = \qquad \{ \text{ def of } M \ \} \\
\qquad \langle \downarrow p : \quad 0 \ \le\ p\ \le\ n : \quad M.p.n \ *\ A.n)\rangle \quad \downarrow \quad 1 \\
\quad = \qquad \{ \text{ case analysis } \ \} \\
\qquad \begin{cases} (\langle \downarrow p : \quad 0 \ \le\ p\ \le\ n : \quad M.p.n\rangle \ *\ A.n) \quad \downarrow \quad 1, & \text{if } 0 \ \le\ A.n \\ (\langle \uparrow p : \quad 0 \ \le\ p\ \le\ n : \quad M.p.n\rangle \ *\ A.n) \quad \downarrow \quad 1, & \text{if } A.n \ \le\ 0 \end{cases} \\
\quad = \qquad \{ \ Ps \ ; \ \ Pt \ \} \\
\qquad \begin{cases} (t \ *\ A.n) \quad \downarrow \quad 1, & \text{if } 0 \le A.n \\ (s \ *\ A.n) \quad \downarrow \quad 1, & \text{if } A.n \le 0 \end{cases} \\
]|
\end{array}
$$

This gives us

$$
\begin{array}{l}
|[\ \textit{Context:} \quad P \quad \wedge \quad n \neq N \\
\quad \{Ps \ \wedge\ Pt\} \\
\qquad \textbf{if} \ \ 0 \ \le\ A.n \quad \rightarrow \quad t \quad := \quad (t \ *\ A.n) \downarrow 1 \\
\qquad [] \ \ A.n \ \le\ 0 \quad \rightarrow \quad t \ := \ (s \ *\ A.n) \downarrow 1 \\
\qquad \textbf{fi} \\
\quad \{(n \ :=\ n+1).Pt\} \\
]|
\end{array}
$$

Our solution is thus

$$
\begin{array}{l}
n, r, s, t \ := \ 0, 1, 1, 1 \\
;\ \textbf{do} \ \ n \neq N \quad \rightarrow \\
\qquad \textbf{if} \ \ 0 \le A.n \quad \rightarrow \quad s, t \qquad := \qquad (s \ *\ A.n) \uparrow 1, \quad (t \ *\ A.n) \downarrow 1 \\
\qquad [] \ A.n \ \le\ 0 \quad \rightarrow \quad s, t \qquad := \qquad (t \ *\ A.n) \uparrow 1, \quad (s \ *\ A.n) \downarrow 1 \\
\qquad \textbf{fi} \\
\qquad ;\ r, n \qquad := \qquad r \uparrow s, \quad n+1 \\
\quad \textbf{od}
\end{array}
$$

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