

## Base X to decimal conversion

We define, for  $0 \leq n \leq N$  :

$$G.n = \langle \Sigma i : n \leq i < N : f.i * X^{i-n} \rangle$$

and our task is to derive a program which computes  $G.0$  .

\* \* \*

We first derive a recurrence relation for  $G$  . We have

$$G.n = 0 \quad \text{if } n = N$$

Moreover we observe

$$\begin{aligned} & G.n \\ = & \{ \text{definition of } G \} \\ & \langle \Sigma i : n \leq i < N : f.i * X^{i-n} \rangle \\ = & \{ \text{split off } i = n, \text{ requires } n < N \} \\ & f.n + \langle \Sigma i : n+1 \leq i < N : f.i * X^{i-n} \rangle \\ = & \{ \text{algebra} \} \\ & f.n + \langle \Sigma i : n+1 \leq i < N : f.i * X^{i-n-1} * X \rangle \\ = & \{ * \text{ over } \Sigma ; \text{arithmetic} \} \\ & f.n + X * \langle \Sigma i : n+1 \leq i < N : f.i * X^{i-(n+1)} \rangle \\ = & \{ \text{definition of } G \} \\ & f.n + X * G.(n+1) \end{aligned}$$

\* \* \*

The postcondition of our program is

$$R : G.0 = r$$

We choose as invariant  $P$  , which is the conjunction of

$$Pn : 0 \leq n \leq N$$

$$Pr : G.0 = r + X^n * G.n$$

which we initially establish with  $r, n := 0, 0$  .

The repetition can terminate whenever  $RHS.R = RHS.Pr$  :

$$\begin{aligned}
 r &= r + X^n * G.n \\
 &= \{ \text{algebra} \} \\
 0 &= X^n * G.n \\
 \Leftarrow &\{ \text{property of } G \} \\
 n &= N
 \end{aligned}$$

and hence  $n \neq N$  is an acceptable guide. We are heading for a program of the form

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r, n := 0, 0
; do n ≠ N →
    "Increase n under invariance of P"
od

```

\* \* \*

We observe

$$\begin{aligned}
 \llbracket \text{Context: } P \wedge n \neq N \\
 r + X^n * G.n \\
 &= \{ \text{property of } G, n < N \} \\
 r + X^n * (f.n + X * G.(n+1)) \\
 &= \{ \text{algebra} \} \\
 r + X^n * f.n + X^{n+1} * G.(n+1) \\
 \rrbracket
 \end{aligned}$$

hence our solution is

```

r, n := 0, 0
; do n ≠ N →
    r, n := r + X^n * f.n, n + 1
od

```

The proofs of termination and maintenance of  $Pn$  under  $n := n + 1$  are standard hence we omit them.

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