fusc

In *Programming: The Derivation of Algorithms* Anne Kaldewaij gives us the recurrence

$fusc.0 \ = \ 0, \ fusc.1 \ = \ 1$

$fusc.(2*n) \ = \ fusc.n$

$fusc.(2*n \ + \ 1) \quad = \quad fusc.n \ + \ fusc.(n+1) \quad \text{for} \quad n \geq 0$

and asks us to derive a program for the computation of $fusc.N$, $N \geq 0$.

* *
*

The postcondition of our program is

$R: \qquad r \ = \ fusc.N$

Our next task is to choose an invariant. The shape of the recurrence suggests as invariant the conjunction of

$\langle \ fusc.n \ + \ fusc.(n+1) \ \rangle$

$Pn: \qquad 0 \leq n \leq N$

where

$\langle \ x \ \rangle \quad \equiv \quad fusc.N \ = \ x$

Kaldewaij hints that we should compute $fusc.78$, so let us do so. We can compute this in either a depth-first or breadth-first manner. We choose the latter as it is simpler (we do not have to store $fusc$ values for later use).

$fusc.78$

$= \qquad \{ \ \}$

$fusc.39$

$= \qquad \{ \ \}$

$fusc.19 \qquad + \qquad fusc.20$

$= \qquad \{ \ \}$

$fusc.9 \qquad + \qquad fusc.10 \qquad + \qquad fusc.20$

$= \qquad \{ \ \}$

$fusc.9 \quad + \quad 2 * fusc.10$

$= \quad \{ \ \}$

$fusc.4 \quad + \quad fusc.5 \quad + \quad 2 * fusc.10$

$= \quad \{ \ \}$

$fusc.4 \quad + \quad 3 * fusc.5$

$= \quad \{ \ \}$

$fusc.2 \quad + \quad 3 * fusc.5$

$= \quad \{ \ \}$

$4 * fusc.2 \quad + \quad 3 * fusc.3$

$= \quad \{ \ \}$

$4 * fusc.1 \quad + \quad 3 * fusc.3$

$= \quad \{ \ \}$

$7 * fusc.1 \quad + \quad 3 * fusc.2$

$= \quad \{ \ \}$

10

This calculation suggests we generalise the invariant to

$P: \quad \langle \, a * fusc.n \quad + \quad b * fusc.(n+1) \, \rangle$

which we initially establish with $n, a, b \ := \ N, 1, 0$. We are heading for a program of the form

$n, a, b \ := \ N, 1, 0$
; **do** $n \neq 0 \quad \rightarrow$
  "Decrease n under invariance of P "
  od

*   *
*

The shape of $fusc$ suggests we investigate two cases: $even.n$ and $odd.n$. We observe

|[*Context:* $even.n$

$a * fusc.n \quad + \quad b * fusc.(n+1)$

$=$ { property of $fusc$; algebra }

$a * fusc.(n \textbf{ div } 2) \quad + \quad b * fusc.((n+1) \textbf{ div } 2) \quad + \quad b * fusc.(((n+1) \textbf{ div } 2) \quad + \quad 1)$

$=$ { $(n+1) \textbf{ div } 2 \ = \ n \textbf{ div } 2$; follows from $odd.(n+1)$ }

$a * fusc.(n \textbf{ div } 2) \quad + \quad b * fusc.(n \textbf{ div } 2) \quad + \quad b * fusc.((n \textbf{ div } 2) \quad + \quad 1)$

$=$ { algebra }

$(a+b) * fusc.(n \textbf{ div } 2) \quad + \quad b * fusc.((n \textbf{ div } 2) \quad + \quad 1)$

$=$ $\triangledown \quad a, n \quad := \quad a+b, \ n \textbf{ div } 2 \quad \triangledown$

$a * fusc.n \quad + \quad b * fusc.(n+1)$

]|

and

|[*Context:* $odd.n$

$a * fusc.n \quad + \quad b * fusc.(n+1)$

$=$ { property of $fusc$; algebra }

$a * fusc.(n \textbf{ div } 2) \quad + \quad a * fusc.((n \textbf{ div } 2) \quad + \quad 1) \quad + \quad b * fusc.((n+1) \textbf{ div } 2)$

$=$ { $(n+1) \textbf{ div } 2 \ = \ (n \textbf{ div } 2)+1$; follows from $even.(n+1)$ }

$a * fusc.(n \textbf{ div } 2) \quad + \quad a * fusc.((n \textbf{ div } 2) \ + \ 1) \quad + \quad b * fusc.((n \textbf{ div } 2)+1)$

$=$ { algebra }

$a * fusc.(n \textbf{ div } 2) \quad + \quad (a+b) * fusc.((n \textbf{ div } 2) \quad + \quad 1)$

$=$ $\triangledown \quad b, n \quad := \quad a+b, \ n \textbf{ div } 2 \quad \triangledown$

$a * fusc.n \quad + \quad b * fusc.(n+1)$

]|

Hence our solution

$$
\begin{array}{l}
\quad n, a, b \ := \ N, 1, 0 \\
; \textbf{do} \quad n \neq 0 \quad \rightarrow \\
\qquad \textbf{if} \quad even.n \quad \rightarrow \quad a, n \quad := \quad a + b, \ n \ \textbf{div} \ 2 \\
\qquad [\,] \ \ odd.n \quad \rightarrow \quad b, n \quad := \quad a + b, \ n \ \textbf{div} \ 2 \\
\qquad \textbf{fi} \\
\quad \textbf{od} \\
; \ r := b
\end{array}
$$

The proofs of termination and maintenance of Pn under $n \ := \ n \ \textbf{div} \ 2$ are standard hence we omit them.

*   *
    *

Acknowledgement

Jeremy Weissmann provided close supervision and invaluable guidance in the composition of this note.

E. Emmanuel Macaulay
eric@mathmeth.com

10 February 2007