Giving more space to high school mathematics

Introduction

In their delightful paper *Doing High School Mathematics Carefully* Ralph-Johan Back and Joakim von Wright apply the calculational style to a variety examples drawn from high school and first-year university mathematics. However they fail to avail themselves of a crucial tool - the use of space to aid in the parsing of formulae. We will remedy that here in our presentation of Back and Wright's proofs. The reader is urged to compare the derivations below with their original presentations. To that end the page number where Back and Wright's original presentation can be found is included.

Example 0 (p3)

$$
\begin{array}{ll}
 & (x+1)^2 = 0 \\
= & \qquad \{ \text{ rule: } \langle \forall a : : \; a^2 = 0 \quad \equiv \quad a = 0 \rangle \; \} \\
 & x + 1 = 0
\end{array}
$$

The indentation of the hint makes it easier to skip it if the reader so wishes.

Example 1 (p4)

$$
\begin{array}{ll}
 & x * (x-2) \;\; = \;\; 3 * (x-2) \\
= & \qquad \{ \text{ subtract right-hand side from both sides } \} \\
 & x * (x-2) \;\; - \;\; 3 * (x-2) \quad = \quad 0 \\
= & \qquad \{ \text{ distributivity } \} \\
 & (x-3) * (x-2) \\
= & \qquad \{ \text{ rule: } a * b = 0 \quad \equiv \quad a = 0 \; \vee \; b = 0 \; \} \\
 & x - 3 = 0 \; \vee \; x - 2 = 0 \\
= & \qquad \{ \text{ add 3 to both sides in left disjunct } \} \\
 & x = 3 \; \vee \; x - 2 = 0 \\
= & \qquad \{ \text{ add 2 to both sides in right disjunct } \} \\
 & x = 3 \; \vee \; x = 2
\end{array}
$$

Example 2 (p5)

$4 * x^2 \ + \ 3 * x = 0$

$= \qquad \{ \text{ distributivity } \}$

$x * (4 * x \ + \ 3) \ = \ 0$

$= \qquad \{ \text{ rule: } a * b = 0 \quad \equiv \quad a = 0 \ \vee \ b = 0 \ \}$

$x = 0 \quad \vee \quad 4 * x \ + \ 3 \quad = \quad 0$

$= \qquad \{ \text{ arithmetic } \}$

$x = 0 \quad \vee \quad x \ = \ -3/4$

$4 * x^2 \ + \ 3 \quad = \quad 0$

$= \qquad \{ \text{ subtract 3 from both sides } \}$

$4 * x^2 \ = \ -3$

$= \qquad \{ \text{ divide both sides by 4 } \}$

$x^2 \ = \ -3/4$

$= \qquad \{ \ x \text{ is real and no square of a real is negative } \}$

$false$

$4 * x^2 \ - \ 3 \quad = \quad 0$

$= \qquad \{ \text{ add 3 to both sides } \}$

$4 * x^2 \ = \ 3$

$= \qquad \{ \text{ divide both sides by 4 } \}$

$x^2 \ = \ 3/4$

$= \qquad \{ \ a^2 = b \quad \equiv \quad a = \sqrt{b} \quad \vee \quad a = -\sqrt{b} \ \}$

$x \ = \ \sqrt{3/4} \quad \vee \quad x \ = \ -\sqrt{3/4}$

$= \qquad \{ \text{ arithmetic } \}$

$x \quad = \quad \sqrt{3} \ / \ 2 \qquad \vee \qquad x \quad = \quad -\sqrt{3} \ / \ 2$

Example 3 (p7)

$f.x$ is defined

$=$ { definition of f }

$\frac{1}{x^2-1}$ is defined

$=$ { known rules about definedness of rational expressions }

$x^2 - 1 \;\neq\; 0$

$=$ { def. of $\neq$ }

$\neg(x^2 - 1 \;=\; 0)$

$=$ { solutions of $x^2 - 1 \;=\; 0$ }

  $x^2 - 1 \;=\; 0$

  $=$ { difference of two squares }

  $(x+1) * (x-1) \;=\; 0$

  $=$ { zero product }

  $x \;=\; -1 \quad \vee \quad x = 1$

• $\neg(x \;=\; -1 \quad \vee \quad x = 1)$

$=$ { DeMorgan }

$\neg x \;=\; -1 \quad \wedge \quad \neg x \;=\; 1$

$=$ { definition of $\neq$ }

$x \;\neq\; -1 \quad \wedge \quad x \neq 1$

Example 4 (p9)

$p \Rightarrow (p \wedge q \;\Rightarrow\; q)$

$=$ { replace sub-term $p \wedge q \;\Rightarrow\; q$ }

  $|[\, p$

  $\triangleright \quad p \wedge q \;\Rightarrow\; q$

  $=$ { replace equals for equals using local assumption $p \equiv \mathit{true}$ }

  $\mathit{true} \wedge q \;\Rightarrow\; q$

  $=$ { identity of $\wedge$ }

  $q \Rightarrow q$

  $=$ { reflexivity }

  true

• $p \Rightarrow \mathit{true}$

$=$ { right zero of $\Rightarrow$ }

true

Example 5 (p9)

$(x-a)^2+b$ attains its smallest value

$=$ { translation into formal notation }

$\langle \forall y :: (x-a)^2+b \quad \leq \quad (y-a)^2+b \rangle$

$=$ { subtract b from both sides }

$\langle \forall y :: (x-a)^2 \ \leq \ (y-a)^2 \rangle$

$=$ { range split, $y=a \ \vee \ y \neq a$; one-point rule }

$(x-a)^2 \leq 0 \quad \wedge \quad \langle \forall y : \ y \neq a : \ (x-a)^2 \ \leq \ (y-a)^2 \rangle$

$=$ { simplify left conjunct }

- $(x-a)^2 \leq 0$

  $=$ { squares are always non-negative }

  $(x-a)^2 = 0$

  $=$ { property of square }

  $x - a \ = \ 0$

  $=$ { add a to both sides }

  $x = a$

- $x = a \ \wedge \ \langle \forall y : \ y \neq a : \ (x-a)^2 \ \leq \ (y-a)^2 \rangle$

$=$ { simplify term of right conjunct }

- $|[\ x = a \ ; \ y \neq a$

  $\triangleright \quad (x-a)^2 \ \leq \ (y-a)^2$

  $=$ { substitute using assumption }

  $0 \ \leq \ (y-a)^2$

  $=$ { squares are always non-negative }

  true

- $x = a \ \wedge \ true$

$=$ { identity of $\wedge$ }

$x = a$

$x^2 - x - 6$ attains its smallest value

$=$ { make expression into a square }

- $x^2 - x - 6$

  $=$ { manipulate middle term }

  $x^2 \ - \ 2 * x * (1/2) \ - \ 6$

  $=$ { manipulate last term }

  $x^2 \ - \ 2 * x * (1/2) \ + \ (1/2)^2 \ - \ (1/2)^2 \ - \ 6$

$=$ { write as square }

$(x - 1/2)^2 - 25/4$

• $(x - 1/2)^2 - 25/4$ attains its smallest value

$=$ { preceding derivation }

$x = 1/2$

Example 6 (p12)

f is uniformly continuous

$=$ { definition }

$\langle \forall \epsilon : \epsilon > 0 : \langle \exists \delta : \delta > 0 : \langle \forall x, y : x - y < \delta \quad \wedge \quad x > y : |2x - 2y| < \epsilon \rangle\rangle\rangle$

$\Leftarrow$ { focus on term, replace in monotonic context }

•|[$\epsilon > 0$

$\triangleright$ $\langle \exists \delta : \delta > 0 : \langle \forall x, y : x - y < \delta \quad \wedge \quad x > y : |2x - 2y| < \epsilon \rangle\rangle$

$\Leftarrow$ { focus on term of $\forall$, replace in monotonic context }

•|[$\delta > 0 \quad x - y < \delta \quad x > y$

$\triangleright$ $|2x - 2y| < \epsilon$

$=$ { simplify using assumptions }

$x - y < \epsilon/2$

$\Leftarrow$ { transitivity using assumptions }

$\delta \leq \epsilon/2$

• $\langle \exists \delta : \delta > 0 : \langle \forall x, y : x - y < \delta \quad \wedge \quad x > y : \delta \leq \epsilon/2 \rangle\rangle$

$\Leftarrow$ { $\exists$-introduction rule, witness for δ is $\epsilon/2$ }

$\delta > 0 \wedge \langle \forall x, y : x - y < \delta \quad \wedge \quad x > y : \epsilon/2 \leq \epsilon/2 \rangle$

$=$ { simplify using assumptions }

$true$

• $\langle \forall \epsilon : \epsilon > 0 : true \rangle$

$=$ { predicate calculus }

$true$

Example 7 (p13)

$\langle \forall n :\ n > 0 :\ \Phi_n \rangle$

$=$ { induction on n }

1. $f.\langle \Sigma\, k : 1 \le k \le 1 : \Phi_k x_k \rangle$

$=$ { simplify summation, using assumption $\Phi_1 = 1$ }

$f.(x_1)$

$=$ { introduce trivial summation, use assumption $\Phi_1 = 1$ }

$\langle \Sigma\, k : 1 \le k \le 1 : f.(x_k) \rangle$

2. $|[\ \Phi_{n-1} \quad ; n > 1$

$\triangleright$ $f.\langle \Sigma\, k : 1 \le k \le n : \Phi_k x_k \rangle$

$=$ { split off $k = n$ }

$f.(\langle \Sigma\, k : 1 \le k \le n-1 : \Phi_k x_k \rangle$

$=$ { rewrite left summand }

$f.((1-\Phi_n) * \langle \Sigma\, k : 1 \le k \le n-1 : \frac{\Phi_n}{1-\Phi_n} x_k + \Phi_n x_n \rangle$

$\le$ {f convex, $0 \le \Phi_n \le 1$}

$(1-\Phi_n) * f.\langle \Sigma\, k : 1 \le k \le n-1 : \frac{\Phi_n}{1-\Phi_n} x_k \rangle + \Phi_n * f.(x_n)$

$\le$ {replace subcomponent in monotonic context}

- $f.\langle \Sigma\, k : 1 \le k \le n-1 : \frac{\Phi_n}{1-\Phi_n} x_k \rangle$

  $\le$ {use induction assumption Φ_{n-1} with $\Phi_k := \frac{\Phi_k}{1-\Phi_n}$}

  $\langle \Sigma\, k : 1 \le k \le n-1 : \frac{\Phi_k}{1-\Phi_n} * f.(x_k) \rangle$

- $(1-\Phi_n) * \langle \Sigma\, k : 1 \le k \le n-1 : \frac{\Phi_k}{1-\Phi_n} * f.(x_k) \rangle + \Phi_n * f.(x_n)$

$\le$ {simplify}

$\langle \Sigma\, k : 1 \le k \le n : \Phi_k * f.(x_k) \rangle$

- *true*

Birmingham, 10 October 2006