We are asked to show for integer $n \geq 0$

(0) $\quad \sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}$

Before proceeding any further we would do well to express this sum in a notation more amenable to manipulation.

Consider the formula $\binom{n}{k}$. This is an asymmetric function of the sum of two arguments and one of the two arguments. We observe

$$\begin{aligned} & \binom{n}{k} \\ = & \quad \{n, k := i+j, j\} \\ & \binom{i+j}{j} \\ = & \quad \{\text{definition of } \binom{n}{k}\} \\ & (i+j)! \;/\; i! \,*\, j! \end{aligned}$$

We define

(1) $\quad P.i.j \quad = \quad (i+j)! \;/\; i! * !j!$

We have

(2) $\quad P.i.j \;=\; P.j.i$

(3) $\quad P.i.j \quad = \quad (i+j)/j \;*\; P.i.(j-1)$

(4) $\quad j \,*\, P.i.j \quad = \quad (i+j) \,*\, P.i.(j-1)$

(5) $\quad i \,*\, P.i.j \quad = \quad (i+j) \,*\, P.(i-1).j$

(6) $\quad P.i.j \quad = \quad P.(i-1).j \;+\; P.i.(j-1)$

and (0) becomes the much nicer

(7) $\quad P.(i+1).j \quad = \quad \langle \Sigma k : 0 \leq k \leq j : P.i.j \rangle$

Since our formulae are in terms of $P.i.j$ we perform the substitution $i := i-1$ to get

(8) $\quad P.i.j \quad = \quad \langle \Sigma k : 0 \leq k \leq j : P.(i-1).k \rangle$

and it is this that we shall prove, by induction on j.

Base case

$$\begin{aligned} & P.i.0 \quad = \quad \langle \Sigma k : 0 \leq k \leq 0 : P.(i-1).k \rangle \\ = & \quad \{\text{definition of } P.i.j, \text{ empty range}\} \\ & i! \;/\; i! \quad = \quad 1 \\ = & \quad \{\text{algebra}\} \\ & \mathit{true} \end{aligned}$$

Inductive step

$\langle \Sigma k : 0 \leq k \leq j+1 : P.(i-1).k \rangle$
$=$ {split off term}
$\langle \Sigma k : 0 \leq k \leq j : P.(i-1).k \rangle + P.(i-1).(j+1)$
$=$ {induction hypothesis}
$P.i.j \ + \ P.(i-1).(j+1)$
$=$ {(6) with $j := j+1$}
$P.i.(j+1)$

And we're done.

[0] Edsger Dijkstra, A stupid notation, EWD782 .

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