

The existence of a particular subsequence

Given integer array $A[0..N)$, $1 \leq N$, we wish to develop a program establishing postcondition

$$R: \quad b \equiv G.N$$

where, for $0 \leq n \leq N$:

$$G.n \equiv \langle \exists p, q : 0 \leq p < q < n : A.p - A.q \leq 2 \rangle$$

$$\begin{array}{ccc} & * & * \\ & & * \end{array}$$

We are heading for a program of the form

```

n := 0; b := false;
{inv: b ≡ G.n}
do n ≠ N →
    “Establish b ≡ G.(n + 1)”
    ;n := n + 1
od
{b ≡ G.N}

```

With regards to termination we appeal to the well-known

Lemma

For constant N , $0 \leq N$, the following program terminates:

```

n := 0
;do n ≠ N → ... n := n + 1 od

```

where clearly ... does not contain assignments to n .

End Lemma.

$$\begin{array}{ccc} & * & * \\ & & * \end{array}$$

In order to refine “Establish $b \equiv G.(n + 1)$ ” we try to derive a simple recurrence relation for G :

$$\begin{aligned}
& G.(n+1) \\
= & \{ G \} \\
& \langle \exists p, q : 0 \leq p < q < n+1 : A.p - A.q \leq 2 \rangle \\
= & \{ \text{split off } q = n \} \\
& \langle \exists p, q : 0 \leq p < q < n : A.p - A.q \leq 2 \rangle \vee \langle \exists p : 0 \leq p < n : A.p - A.n \leq 2 \rangle \\
= & \{ G \} \\
& G.n \vee \langle \exists p : 0 \leq p < n : A.p - A.n \leq 2 \rangle \\
= & \{ \text{see sidebar calculation} \} \\
& G.n \vee \langle \downarrow p : 0 \leq p < n : A.p \rangle - A.n \leq 2 \\
= & \{ \text{introducing } H.n = \langle \downarrow p : 0 \leq p < n : A.p \rangle \} \\
& G.n \vee H.n - A.n \leq 2
\end{aligned}$$

We strengthen the invariant with

$$c = H.n$$

and hence find for “Establish $b \equiv G.(n+1)$ ”

$$\begin{aligned}
b & := b \vee c - A.n \leq 2; \\
& \text{“Establish } c = H.(n+1)\text{”} .
\end{aligned}$$

Sidebar calculation

$$\begin{aligned}
& \langle \exists p : 0 \leq p < n : A.p - A.n \leq 2 \rangle \\
= & \{ \text{lattice theory} \} \\
& \langle \downarrow p : 0 \leq p < n : A.p - A.n \rangle \leq 2 \\
= & \{ + \text{ over } \downarrow \} \\
& \langle \downarrow p : 0 \leq p < n : A.p \rangle - A.n \leq 2
\end{aligned}$$

End of Sidebar calculation.

* *

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With the observations

$$G.0 \equiv \text{false}$$

$H.0 = \infty$

$H.(n + 1) = H.n \downarrow A.n$

we may collect the pieces to arrive at our solution:

```
n := 0; b := false; c := ∞;
{inv: (b ≡ G.n) ∧ c = H.n}
do n ≠ N →
    b := b ∨ c - A.n ≤ 2
    ;c := c ↓ A.n
    ;n := n + 1
od
{b ≡ G.N}
```

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We can say a bit more, however. As *true* is a fixpoint of \vee we may weaken the termination condition to $n = N \vee b$. Our guard is now $n \neq N \wedge \neg b$ so $b \equiv \text{false}$ holds within the loop allowing the simplification of the assignment to b . Thus we arrive at our final program

```
n := 0; b := false; c := ∞;
{inv: (b ≡ G.n) ∧ c = H.n}
do n ≠ N ∧ ¬ b →
    b := c - A.n ≤ 2
    ;c := c ↓ A.n
    ;n := n + 1
od
{b ≡ G.N}
```

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