Mr Mayer,

Stumbling across [0] I was somewhat taken aback to see your claim that ‘It is not acceptable to start with what you want to prove and then end up with a true statement.’ Consider the following. (In the sequel . denotes function application, and * denotes multiplication.)

Given

(0) $sin.(x+t) \;=\; sin.x * cos.t \;+\; cos.x * sin.t$

(1) $cos.(x+t) \;=\; cos.x * cos.t \;-\; sin.x * sin.t$

(2) $sin.(\pi/4) \;=\; \sqrt{2}/2$

(3) $cos.(\pi/4) \;=\; \sqrt{2}/2$

We observe

$$sin.(x+\pi/4) \;+\; cos.(x+\pi/4) \;=\; \sqrt{2} * cos.x$$

= { (0), (1) }

$$sin.x * cos.(\pi/4) \;+\; cos.x * sin.(\pi/4) \;+\; cos.x * cos.(\pi/4) \;-\; sin.x * sin(\pi/4) \;=\; \sqrt{2} * cos.x$$

= { (2), (3) }

$$sin.x * \sqrt{2}/2 \;+\; cos.x * \sqrt{2}/2 \;+\; cos.x * \sqrt{2}/2 \;-\; sin.x * \sqrt{2}/2 \;=\; \sqrt{2} * cos.x$$

= { algebra }

$$\sqrt{2} * cos.x \;=\; \sqrt{2} * cos.x$$

= { reflexivity of = }

true

This is effectively the proof that the student you complained about gave, though it has been formatted differently. Note that contrary to what you wrote in your comment we have easily shown that every step is reversible - the first column contains =, an operator we know to be symmetric.

The problem with the above proof is not that we start with what we want to prove per se, but that we keep dragging the $\sqrt{2} * cos.x$ through the proof. A cleaner version would be

$$sin.(x+\pi/4) \;+\; cos.(x+\pi/4)$$

= { (0), (1) }

$$sin.x * cos.(\pi/4) \;+\; cos.x * sin.(\pi/4) \;+\; cos.x * cos.(\pi/4) \;-\; sin.x * sin(\pi/4)$$

= { (2), (3) }

$$sin.x * \sqrt{2}/2 \;+\; cos.x * \sqrt{2}/2 \;+\; cos.x * \sqrt{2}/2 \;-\; sin.x * \sqrt{2}/2$$

= { algebra }

$$\sqrt{2} * cos.x$$

As a nice side effect our proof is shorter.

You complained that when starting with what they want to prove is that students may end up presenting 'one-way' proofs when they in fact have to provide 'two-way' proofs i.e. they prove $X \Rightarrow Y$ instead of $X \equiv Y$. But the problem is not what they start with but that they lack a proof format which allows them to be explicit about the relations which connect the steps of their proof.

In reply to a comment you argue that a student risks producing wrong arguments like '$-1 = 1$ so $(-1)^2 = 1^2$ and thus $1 = 1$ which is true' if they start with the demonstrandum. But your example is far from convincing. Why wouldn't the student just write

$$
\begin{array}{ll}
 & -1 = 1 \\
= & \quad \{ \text{ integers } \} \\
 & false
\end{array}
$$

?

In fact, your injunction that they should never start with what they want to prove is harmful as a possible avenue for problem solving is unnecessarily denied them. There are plenty of theorems for which a perfectly sound proof strategy is not to transform one side into the other but to calculate that the entire expression is equivalent to *true*. For example, let us prove $u - 2^n \quad \approx \quad u$, where $\approx$ is congruence modulo 2^n. In what follows $\sqsubseteq$ means 'divides'.

$$
\begin{array}{ll}
 & u - 2^n \quad \approx \quad u \\
= & \quad \{ \text{ definition of } \approx \} \\
 & 2^n \quad \sqsubseteq \quad (u - (u - 2^n)) \\
= & \quad \{ \text{ arithmetic } \} \\
 & 2^n \sqsubseteq 2^n \\
= & \quad \{ \text{ reflexivity } \} \\
 & true
\end{array}
$$

In the comments section you also say that

'Starting with $\sqrt{2}\cos x = \sqrt{2}\cos x$ and ending with the final result is a proof, but that's not what they wrote.'

But how on earth would the student have known to start with $\sqrt{2} * cos.x \quad = \quad \sqrt{2} * cos.x$? Only if they started with what they wanted to prove in the first place! Why, then, pretend that they didn't?

[0] Steve Mayer, Proof and Logic,
Mathematics weblog (24 October 2004) http://sixthform.info/maths/index.php?m=200410

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