Throughout my life, my two greatest passions have been mathematics and education, two fields which are, in essence, about asking and answering the question "Why?" . I combined these passions recently in my Fulbright research in the Netherlands, where I explored a new, highly effective approach to teaching mathematics. In the process, I discovered that mathematics has a crucial role to play in the wider domain of education: *doing mathematics is the best way to learn to think*. Through my NSF project, I aim to explore this approach further, building on my rich background in mathematics and education to become a finer mathematician and more effective teacher.

My love and talent for mathematics began at an early age. In high school, I scored first in my school on the American Mathematics Contest and took courses at a community college; in college, I took upper-level math courses from my first year, and graduate-level courses by my junior year. Through the Budapest Semesters in Mathematics program, I studied abroad for a year, then returned to Northwestern, receiving the Robert R. Welland Prize for Outstanding Achievement in Mathematics by a Graduating Senior. I graduated cum laude with honors in mathematics, having written a thesis in mathematical logic.

My passion for education began even earlier, in the fourth grade, when my teacher let me give occasional lessons to my peers. I found early on that I had a natural ability not only to explain mathematics, but also to communicate the exhilaration of mathematical exploration. Throughout middle school and high school, I fell naturally into the role of tutor and mentor, eventually becoming aide to the math department, teaching full lessons in algebra, trigonometry, and calculus classes. In college, I was hired by the math department as an assistant for courses in mathematical logic and real analysis. The most pivotal teaching experience during this time was for a course called "Introduction to Proof" : conveying the raw tools of thought was thrilling, and is still a central goal of my teaching today.

Midway through college, I became frustrated with the lack of methodology in my mathematics curriculum. I was being taught many existing techniques, proofs, and theories, but no explicit methods of designing my own. Often a presented proof would amount to a string of mathematical tricks plucked from thin air, like the proverbial rabbit out of the magician's hat. This intuition-based approach could serve as basis for neither a working methodology nor a pedagogy of mathematics, and so as mathematician and teacher alike I was sorely disappointed.

This all changed upon reading the doctoral thesis of Netty van Gasteren. In this revolutionary work, she argues that proofs can be designed constructively from their specifications, using formal, calculational techniques. My first opportunity to experiment with these techniques was in a senior seminar I led in graph theory. Many published proofs and expositions in graph theory rely on tricks and special insights, and in our seminar we explored the extent to which we could avoid these. We created recastings of definitions that were clearer and more usable, and designed new proofs that were "rabbit-free" , while still crisp and elegant. The success of this seminar was the first indication that calculational mathematics could spark a revolution in mathematics pedagogy.

Through a Fulbright fellowship, I continued to research the calculational method with Wim Feijen, van Gasteren's mentor and collaborator. I improved on my graph theory results, and also investigated a calculational pedagogy of mathematics. To my surprise, I discovered that the calculational method is an excellent tool for teaching not only mathematics, but also thought in general: Calculation itself requires only simple hand-eye coordination, so all one's energies can be devoted to perfecting the raw skills of effective reasoning, skills which are applicable to any field of inquiry. Indeed, the skills I have developed through my research have not only made me a better scientist, but a better artist as well. For example, I write poetry consciously, by first choosing my themes, then finding phonetic sounds which I feel evoke them, then developing a rhythmic scheme for organizing the sounds, and finally choosing words to implement them. I compose cryptic crossword clues similarly, creating new words from the puzzle's letters, using wordplay. And when I practice the piano, I pay attention to the unfortunate patterns my hands and arms fall into, the notes or passages I continually miss, and then work consciously to correct these patterns through discipline.

It was in the Netherlands that I had my first opportunities to share these ideas on a broader scale, through my many conversations with Dutch grade schoolers, university students, professors, scientists, poets, and musicians. Despite the language barrier and despite the wide range of ages, interests, and backgrounds, I was able to explain how mathematics could help each of them to become better thinkers. And as I did, I saw sparks of excitement flash within them: they had been taught to wait for new ideas to "come to them" ; now they saw that these ideas could be designed at will. Learning to think more effectively is thrilling and empowering for everyone, from any background, from any culture, of any age.

This year I am teaching 6th and 7th graders at the Los Angeles Center for Enriched Studies, a public magnet school whose student body is primarily Latino and African-American. I begin each class with syntactic exercises in copying, substitution, rearrangement, and distributivity, to help develop my students' hand-eye coordination and handwriting skills. Through these simple exercises, my students develop a comfort and fluency in syntactic manipulation, but also a great deal of discipline: They learn to be careful and cautious, and have acquired a certain respect for thinking and writing. Though some of my efforts have certainly met with failure, the successes have convinced me that the capacity for rigorous, abstract thought can be taught explicitly to children from a very early age, through calculational mathematics.

Ultimately I wish to realize my pedagogical ideas in my own school. At the center of the curriculum will be a class on thinking, where a student will first become aware that they are in possession of a powerful tool —their mind— that can be harnessed and put to use solving problems in a disciplined fashion. Through my mentoring, they will become more cautious thinkers, wary of the traps of thought, in particular the vagueness of natural language. They will learn constructive methods of abstracting "real-world" problems to make them more amenable to rational or mathematical thought, and they will understand how they can make these abstractions themselves, not by intuition, but by design. Along the way, they will apply their developing skills to a variety of subjects, from music, to literature, to science, to sport, so that their abilities of creative expression and rational thought can develop in tandem.

My teaching experiences in college, in the Netherlands, and now in my classroom at LACES have been truly inspirational, and have sharpened my desire to distill and develop this methodology through graduate study. Graduate research will represent a unique and critical point in my evolution as scientist and teacher, where the ideas developed through my years of experience in research and mathematics will fully crystallize, forming a solid foundation from which I can strike out with conviction and resolve, an ideological foundation that will enable me to teach and communicate far more clearly, effectively, and widely than ever before.

As an undergraduate, I served two summers as linguistics research assistant at Northwestern University. The first summer, we conducted an in-depth study of how 'comparison constructions' work in particular languages; my language was Hungarian. The research pattern took place in weekly cycles, as follows: First, I would read some of the literature on linguistic comparison. Then, I would formulate questions based on this literature, which I would take to my three Hungarian interviewees. With their raw responses in hand, I would compare and contrast my findings with those of another research assistant, who was working with Mandarin Chinese. Finally, we would present our conclusions to Professor Kennedy, and discuss. As a product of our discussions, we would have new readings and questions, to begin the process anew. The second summer, we embarked on a cross-linguistic study of particular comparison constructions. I was assigned to a particular construction, and a group of 26 languages. Similar to the previous year, my task was to find as much relevant information about these languages as I could, and then present it to the professor.

From this research, I gained experience in broad surveys of literature, as well as in the iterative nature of research: Results are obtained, but then refined, refined, and refined again.

Also as an undergraduate, I wrote a senior thesis in mathematical logic. My goal was to present a modern simplification of a severely entangled proof by Alfred Tarski. First, I read the original article by Tarski in stages, presenting and discussing weekly with my advisor. I also read supplementary material about quantifier elimination, the specific technique Tarski used in his proof. Then I worked to clarify the logical structure of his proof, sometimes inverting whole sections, or recasting arguments in crisper terminology. I wrote up my results as a paper, which was refereed by my advisor and another professor in the mathematics department. As a result, I graduated with honors in mathematics.

This was my first brush with intense mathematical research. I not only learned how to better structure proofs using Tarski's technique —which is prevalent in the field— , but also gained experience in approaching and analyzing primary literature, and in writing and structuring a large-scale mathematical paper.

After graduating, I spent a year as a graduate student at UC Santa Cruz, undertaking some independent research in linguistics. My major research project was a series of papers in the phonological framework of Optimality Theory . The first paper was based on an article I had read in class on the phenomenon of 'ternary rhythm' ; I critiqued this article and provided the beginnings of a new analysis. Through collaboration with my professor and the author of the article, I produced a second paper fleshing out my analysis in more detail, and also describing and applying an entirely new method of constructing phonological analyses by mathematical proof. This new level of rigor uncovered that, at least with respect to ternary rhythm, the modern framework of Optimality Theory itself faced crucial problems. The following quarter I refined this argument further into a third paper, which I submitted for publication to the journal *Phonology* .

The experience was exhilarating. As with my previous research experiences, again I learned how to more effectively use the literature to inform my analyses, and to use collaboration and discussion to refine them. But never before did this take place on such a large scale, or with so many theoretical discoveries and innovations spurring me onward.

Following my year in Santa Cruz, I received a Fulbright fellowship to research mathematics and mathematics pedagogy in the Netherlands. My proposal was to lay the groundwork for a calculational style of reasoning in graph theory, and for a pedagogy of the calculational mathematical style in general. (Here 'calculational' refers to the kind of symbolic manipulation we use when we multiply two numbers, or solve algebra problems.) I focussed on the former goal in the first months. I showed that by allowing edges to be pairs of some superset of the vertex set, the mathematical notion of graph could be generalized in order to simplify many existing proofs in the literature. I designed formalisms exploiting this generalization which were more elegant and usable than their counterparts in the literature. I also explored how algorithmic techniques could be used to prove theorems of graph theory, firstly through the notion of algorithmic invariants, and secondly through the notion of program inversion. Towards the latter goal of pedagogy, I extended and homogenized the existing calculational methods (for instance by allowing the context of calculations to be nonscalar), making them more powerful, and yet simpler to teach and use. I also designed a conceptual interface with which to talk elegantly about ideas, concepts, mathematics, and abstraction. And though it was not part of my proposal explicitly, I did much more in Eindhoven: for example, I learned the technique of programming by derivation, co-created our website, http://mathmeth.com , and applied the calculational style to problems from abstract algebra, logic, number theory, and other areas of discrete math.

All this work was conducted through many intense hours of thought; in-depth consultation of our field's background literature, as well as the wider mathematical literature; and collaboration and discussion with my two student colleagues, as well as several professors from our department. Another excellent means of collaboration was the Tuesday Afternoon Club, where each word of a new document would be read slowly and out loud, scrutinized by often-ruthless professors and colleagues. The quality of work emerging from this trial-by-fire would be several times greater than the quality of work entering it. Naturally, in this rich intellectual environment, I was highly productive, writing over 30 technical notes, starting about 15 more, and collaborating almost daily with my colleagues on their projects. And even though my research was in a sense open-ended, I was more productive than I have ever been in my life: while on vacation in Zürich, I wrote a technical note on napkins at Starbucks; while returning to the US to teach at Calculus Camp, I wrote while in my sleeping bag; even at the end, after my farewell party in Eindhoven, two days before my flight back to the US, I wrote two notes in white heat, one showing how a calculational algebraic technique could be applied to group strategy problems, and another detailing an elegant and innovative extension to our proof format.

When you find what you want to do, and find a charged environment in which to do it, research becomes the air you breathe and the food you eat, without feeling like a burden or obligation. Benefits to the scientific community and society come as natural extensions, through publication and through education. This is the sort of research experience I had in the Netherlands, and is exactly the sort of research I wish to continue doing in my NSF project.

**Title:** Discipline of Thought and Calculational Mathematics
**Keywords:** *education, discipline, formalism, mathematics, methodology, pedagogy, thought*

In her PhD thesis, "On the shape of mathematical arguments" , Dutch computing scientist Netty van Gasteren demonstrated how the presentation of mathematical proofs can be improved by adopting syntactic, calculational techniques. Over the past 20 years, these techniques have been crafted into a powerful and teachable mathematical methodology, whereby proofs can be constructively designed from their specifications, rather than intuited. Studying under Wim Feijen, van Gasteren's long-time collaborator and mentor, I played a role in this development through my Fulbright research in the Netherlands. I wish to continue my research in graduate study at the University of Nottingham with one of their closest colleagues, Roland Backhouse.

In the calculational style of mathematics, one first translates the givens and the demonstrandum into a flexible, powerful notation. Then, by syntactically analyzing the resulting formulae, one designs the chain of formal manipulations leading from one into the other. These are the sorts of manipulations we use so well in differentiating $x^2$ , solving $2x + 1 = 3$ , or computing $346 + 123$ . Naturally, in such a syntactic approach, the choice of notation is no longer a mere matter of style, but often the deciding factor between success and failure. Many notations in the literature are like Roman numerals: a cumbersome shorthand and a pain in the neck to reason with. That being said, the goal of the calculational style is not to try to reduce all problems to symbol manipulation, but rather to use calculation judiciously, wherever it can be fruitfully applied.

Since the calculational techniques were originally developed in response to the complexity of program design, several works have explored calculational programming methodology, for example, Roland Backhouse's "Program Construction" and van Gasteren and Feijen's "On a Method of Multiprogramming" , to name only two. But while the broader mathematical methodology has been applied successfully to diverse areas of mathematics, the details of the methodology itself have only been touched upon in the mathematical literature. I aim to remedy this by conducting an in-depth investigation of the calculational style, synthesizing and extending van Gasteren's research, in particular with my own recent work on pointwise calculation, nonscalar assumptions, types, and Skolemization.

My work will serve three purposes. Firstly, it will fulfill a long-standing need within the calculational community, by culling together the essential ideas and tools of the calculational style. Secondly, it will increase the exposure of our methodology to the wider scientific community. (Despite potential benefits of the calculational style both in scientific research and in pedagogy, it is relatively unknown in the academic world, primarily because there has never been a complete and coherent exposition of the style.) And thirdly, as I discuss below, it will lay the groundwork for a planned book on the pedagogy of mathematics and rational thought.

Nottingham will provide a highly charged environment in which to undertake my graduate research. Their three-year degree program centers exclusively on a doctoral thesis, allowing me to begin my research and collaboration straight away. Wim Feijen has described my proposed advisor, Roland Backhouse, as one of the finest calculational mathematicians in the world, and having corresponded with him for the past two years, I strongly concur. He has written books and articles on the calculational style, led the mathematical programming group at the University of Eindhoven for nine years, and currently leads the Foundations of Programming Group in

Nottingham. I have spent several days at the University, getting to know him and his graduate students personally, and we are all looking forward to a fruitful collaboration. Finally, at Nottingham I would be in a prime physical location to collaborate with my colleagues nearby in England, Germany, and the Netherlands.

For the structure of my investigation, I adapt the approach taken by van Gasteren, which has two threads. In the first thread, I discuss and develop the techniques of the modern calculational style, from the principles of explicitness, naming, interface design, abstraction, and separation of concerns; to techniques of human calculation, our proof format, predicate calculus, lattice theory, and heuristics for the exploitation of properties like distributivity and transitivity. In the second thread, I apply our methodology to several non-trivial problems from the literature, showing how the calculational style not only results in cleaner expositions, but also caters to the construction of solutions by analysis and design.

As mentioned above, I wish to craft my thesis ultimately into a book on pedagogy, showing how the calculational style can assist in developing the raw skills of thought.

The common thread in any human pursuit is the need for creativity and discipline: Creativity helps us design new problems and techniques, while discipline helps us solve the problems efficiently, and exploit the techniques effectively. These skills are highly difficult to master, and calculational mathematics provides the ideal environment for developing them.

When we are just learning to think, the concepts we reason about should be very simple, so we can focus on the reasoning itself. Learning to think by, for instance, solving the problem of world hunger is as futile as learning to swim in a tidal wave. So, even though the real-world concepts we wish to reason about are complex and hazy, it is essential to practice in an idealized domain. Mathematical properties are nearly ideal in this respect, as they are simpler and more easily articulated than nonmathematical ones, and among these, calculational properties are the simplest —because they are formulated in terms of symbol manipulation— , and hence truly ideal.

Calculation is just symbolic manipulation, so in a calculational problem we are able to focus entirely on questions of *how* to manipulate: how to define goals clearly, separate concerns, and form generalizations; how to be explicit, organized, and aware of choices. These skills, which outside of mathematics are fairly vague, can now be concretely understood and explicitly taught in terms of symbols. Thus the calculational method helps foster the skills of creativity and discipline of thought (and a taste for simplicity) which become so crucial as the complexity of problems grows.

The book will follow the outline of my thesis, and will split into two sections: The first section will explain the techniques of effective and efficient thought, the need for discipline, and the role mathematics has to play in developing that discipline, as mentioned above. The second section will give readers the technical background they need to design their own mathematical arguments, including an exposition of the predicate and relational calculi, lattice theory, and algorithm design. Also, this section will contain many fine-grained solutions to mathematical problems, showing the reader how to form a mathematical interface with an English-language problem, how to design appropriate terminology to carry the argument, and how to provide detailed justification and reasoning for each step of a calculation.