Exercise 0 from WF122

In this note I would like to record my thoughts about and efforts at a few predicate calculus exercises from Wim Feijen's WF122 .

(0) $[\ P \wedge (X \equiv Y \equiv Z) \ \equiv\ P \wedge X \ \equiv\ P \wedge Y \ \equiv\ P \wedge Z\ ]$

My first thought is to give a catchword to (0) : it could appropriately be called " $\wedge$ over $\equiv\equiv$ " .

Next, I describe the shapes of the formulae. There is a three-way symmetry between X , Y , and Z , which is nicely reflected in the choice of nomenclature. (Thanks, Wim!) The shape "$P \wedge -$" appears symmetrically in X , Y , and Z ; each such term is an equivaland of the entire expression. The other equivaland is $P \wedge (X \equiv Y \equiv Z)$, and the conjunct $X \equiv Y \equiv Z$ is again symmetric in X , Y , and Z . Note finally that P is a conjunct of each equivaland in (0) .

The next concern is: What can be done? What can be massaged? In (0) , we only have $\wedge$ and $\equiv$. I know a property connecting these, namely:

(a) $[\ Q \wedge (A \equiv B) \ \equiv\ Q \wedge A \ \equiv\ Q \wedge B \ \equiv\ Q\ ]$,

but perhaps this is assuming too much. Nevertheless, let us prove (0) from (a) .

Which expressions in (0) and (a) can we match up? It seems that we cannot help breaking the symmetry between X , Y , and Z , because of the shape of (a) , but surely $Q := P$ is a sensible choice. We arbitrarily exploit the associativity of $\equiv$ to group $X \equiv Y \equiv Z$ as $X \equiv (Y \equiv Z)$, and take $A, B := X, (Y \equiv Z)$.

How to calculate? Doing justice to the symmetry in (0) , we aim for the following proof shape:

$P \wedge (X \equiv Y \equiv Z)$

$\equiv$ { ... }

$P \wedge X \ \equiv\ P \wedge Y \ \equiv\ P \wedge Z$,

which suggests using (a) as a rewrite rule. We push forward experimentally:

$$P \ \wedge\ (X \equiv Y \equiv Z)$$

$\equiv$ { associativity of $\equiv$, as above }

$$P \ \wedge\ (\, X \equiv (Y \equiv Z)\,)$$

$\equiv$ { (a) with $Q, A, B \ := \ P, X, (Y \equiv Z)$ }

$$P \wedge X \quad \equiv \quad P \wedge (Y \equiv Z) \quad \equiv \quad P$$

$\equiv$ { we can exploit (a) again to form subexpressions of our goal, so we do, this time with $Q, A, B \ := \ P, Y, Z$ }

$$P \wedge X \quad \equiv \quad P \wedge Y \quad \equiv \quad P \wedge Z \quad \equiv \quad P \quad \equiv \quad P$$

$\equiv$ { $P \equiv P$ is the unit of $\equiv$ }

$$P \wedge X \quad \equiv \quad P \wedge Y \quad \equiv \quad P \wedge Z \qquad .$$

Completely straightforward, except for some problems with my fountain pen. Possibly our discussion should have started by talking about proof shapes, which would have led us to formulate our goal in terms of creating expressions with the shape "$P \wedge -$" . In any case, (a) makes the calculation of (0) straightforward.

So then what about this (a) ? What if we were not so lucky to know it? As we mentioned above, we still need a way to massage $\wedge$ and $\equiv$ in combination.

We may consider the following property fundamental:

(b) $\quad [\ \ Q \ \vee\ (A \equiv B) \quad \equiv \quad Q \vee A \quad \equiv \quad Q \vee B \ \ ] \qquad ,$

which generalizes immediately to:

(b') $\quad [\ \ Q \ \vee\ (A \equiv B \equiv C) \quad \equiv \quad Q \vee A \quad \equiv \quad Q \vee B \quad \equiv \quad Q \vee C \ \ ] \qquad :$

close, but no cigar!

With (b) in hand, we naturally would like a connection between $\wedge$ and $\vee$. We have:

(c) $\quad [\ \neg(A \wedge B) \quad \equiv \quad \neg A \vee \neg B \ ] \qquad ,$

$\quad [\ \neg(A \vee B) \quad \equiv \quad \neg A \wedge \neg B \ ] \qquad$ and

(d) $\quad [\ \ A \wedge B \quad \equiv \quad A \quad \equiv \quad B \quad \equiv \quad A \vee B \ \ ] \qquad .$

I think (c) is to be dispreferred because it introduces a foreign element, $\neg$. However, it is tantalizing enough to try. As practice, let's explore a simpler calculandum than (0) :

$$
\begin{array}{ll}
 & Q \;\wedge\; (A \equiv B) \\
\not\equiv & \{\ \text{predicate calculus}\ \} \\
 & \neg(\,Q \;\wedge\; (A \equiv B)\,) \\
\equiv & \{\ \text{(c) , ie de Morgan}\ \} \\
 & \neg\,Q \;\vee\; \neg(A \equiv B) \\
\equiv & \{\ \text{predicate calculus, preparing to use (b) , and preserving symmetry}\ \} \\
 & \neg\,Q \;\vee\; (A \equiv B \equiv \mathbf{false}) \\
\equiv & \{\ \text{(b) with } Q, A, B, C \;:=\; \neg\,Q, A, B, \mathbf{false}\ \} \\
 & \neg\,Q \vee A \;\equiv\; \neg\,Q \vee B \;\equiv\; \neg\,Q \vee \mathbf{false} \\
\not\equiv & \{\ \text{predicate calculus}\ \} \\
 & \neg\,Q \vee A \;\equiv\; \neg\,Q \vee B \;\equiv\; Q \qquad .
\end{array}
$$

We got a little stuck. But if we'd massaged $A \equiv B$ into $\neg\,A \equiv \neg\,B$, we'd eventually derive (a) .

Aside: In the above we used $[\,\neg\,P \equiv P \equiv \mathbf{false}\,]$, which just suggested the following adorable calculation:

$$\neg A \ \vee \ \neg B$$

$\equiv$ { $\neg/\mathbf{false}$ }

$$(A \equiv \mathbf{false}) \ \vee \ (B \equiv \mathbf{false})$$

$\equiv$ { $\vee$ over $\equiv$ }

$$A \vee B \ \equiv \ A \vee \mathbf{false} \ \equiv \ B \vee \mathbf{false} \ \equiv \ \mathbf{false} \vee \mathbf{false}$$

$\equiv$ { $\neg/\mathbf{false}$ }

$$A \vee B \ \equiv \ A \ \equiv \ B \ \equiv \ \mathbf{false}$$

$\equiv$ { (d) }

$$A \wedge B \ \equiv \ \mathbf{false}$$

$\equiv$ { $\neg/\mathbf{false}$ }

$$\neg(A \wedge B) \qquad ,$$

which proves half of (c) from (d) ! But I don't know how to prove the other half similarly.

(End of Aside.)

Now let us explore $\wedge/\equiv$ using (d) :

$$Q \ \wedge \ (A \equiv B)$$

$\equiv$ { (d) }

$$Q \ \vee \ (A \equiv B) \ \equiv \ Q \ \equiv \ (A \equiv B)$$

$\equiv$ { $\vee$ over $\equiv$ }

$$Q \vee A \ \equiv \ Q \vee B \ \equiv \ Q \ \equiv \ A \ \equiv \ B$$

$\equiv$ { (d) with $A, B := Q, A$ }

$$Q \wedge A \ \equiv \ Q \vee B \ \equiv \ B$$

$\equiv$ { (d) }

$$Q \wedge A \ \equiv \ Q \wedge B \ \equiv \ Q \qquad .$$

And that is (a) .

*　　　*
*

If we didn't know (d) , we'd have to postulate a connection between $\wedge$ and $\vee$, say:

$$(*) \qquad [\ A \wedge B \ \equiv \ A \vee B \ \equiv \ c \ ] \qquad .$$

The hope is that some nontrivial c solves $(*)$ — but note that there is always the trivial solution $c := (A \wedge B \equiv A \vee B)$.

So we explore:

$$Q \wedge (A \equiv B)$$
$$\equiv \quad \{ \ (*) \ \}$$
$$Q \vee (A \equiv B) \ \equiv \ c$$
$$\equiv \quad \{ \ \vee \text{ over } \equiv \ \}$$
$$Q \vee A \ \equiv \ Q \vee B \ \equiv \ c$$
$$\equiv \quad \{ \ (*) \text{ twice } \}$$
$$(Q \wedge A \ \equiv \ c) \ \equiv \ (Q \wedge B \ \equiv \ c) \ \equiv \ c$$
$$\equiv \quad \{ \text{ properties of } \equiv \ \}$$
$$Q \wedge A \ \equiv \ Q \wedge B \ \equiv \ c \qquad .$$

Well that is encouraging, but it only works if c is constant, and we already know it isn't. So we write:

$$(**) \qquad [\ A \wedge B \ \equiv \ A \vee B \ \equiv \ f.A.B \ ] \qquad ,$$

and try again:

$$Q \wedge (A \equiv B)$$
$$\equiv \quad \{ \ (**) \ \}$$
$$Q \vee (A \equiv B) \ \equiv \ f.Q.(A \equiv B)$$
$$\equiv \quad \{ \ \vee \text{ over } \equiv \ \}$$
$$Q \vee A \ \equiv \ Q \vee B \ \equiv \ f.Q.(A \equiv B)$$
$$\equiv \quad \{ \ (**) \ \}$$
$$Q \wedge A \ \equiv \ Q \wedge B \ \equiv \ f.Q.A \ \equiv \ f.Q.B \ \equiv \ f.Q.(A \equiv B) \qquad .$$

I'm not sure where that gets us, but I'm interested now in deriving f —that is, the Golden Rule— from properties of $\wedge$.

For example, from "intuition" , $\wedge$ should satisfy idempotence and symmetry:

<u>Idempotence</u>

$$A \wedge A \;\equiv\; A$$

$\equiv$ { (**) }

$$A \vee A \;\equiv\; f.A.A \;\equiv\; A$$

$\equiv$ { idempotence of $\vee$ }

$$f.A.A \qquad ,$$

so $[\ f.A.A\ ]$, and:

<u>Symmetry</u>

$$A \wedge B \;\equiv\; B \wedge A$$

$\equiv$ { (**) }

$$A \vee B \;\equiv\; B \vee A \;\equiv\; f.A.B \;\equiv\; f.B.A$$

$\equiv$ { symmetry of $\vee$ }

$$f.A.B \;\equiv\; f.B.A \qquad ,$$

so $[\ f.A.B \;\equiv\; f.B.A\ ]$, so f is symmetric in its arguments. This suggests:

- $[\ f.A.B \;\equiv\; A \equiv B\ ]$

as a simple implementation. So this gives us the Golden Rule, and also ensures that idempotence and symmetry of $\wedge$ follow from the unit property and symmetry of $\equiv$.

[Oh, what about the following properties of $\wedge$:

$$A \wedge \textbf{true} \;\equiv\; A$$

$\equiv$ { (**) }

$$A \vee \textbf{true} \;\equiv\; A \;\equiv\; f.A.\textbf{true}$$

$\equiv$ { predicate calculus }

$$A \;\equiv\; f.A.\textbf{true}$$

and:

$$A \wedge \textbf{false} \;\equiv\; \textbf{false}$$

$\equiv$ { (**) }

$$A \vee \textbf{false} \;\equiv\; \textbf{false} \;\equiv\; f.A.\textbf{false}$$

$\equiv$ { predicate calculus }

$$\neg A \;\equiv\; f.A.\textbf{false} \qquad ! \qquad]$$

What would now convince us that this was a useful implementation of $\wedge$? We could check associativity, etc. This is done in PCPS.

*   *
*

This was a very nice opportunity to think.

The Hungarian Pastry Shop, 9 September 2009

Commentary: I am finally typing up this note two months later. It was my first effort at disciplined thought in a long, long time, and it is encouraging and amusing to re-read. Amusing, because it is largely incoherent; there is no expository flow, but only a leisurely ramble of observations. And yet, it is encouraging to see how strict I was at times, filling in a lot of details that I could have left out, because I recognized the importance of being strict when one is trying to take the reins of one's own mind.

Maybe I should have phrased it the other way: I knew that strictness was important, but although I tried to be strict, as can be seen by some of my methodical proofs, I was not strict at all when it came to formulating my thoughts in a coherent way. That is a sobering reflection, because I honestly thought I was crafting a somewhat decent exposition! That I can even follow the note at all now, two months later, is testimony to the fact that all those years of disciplining myself have had a lasting effect. (Thank god!) But clearly there's a lot of work to do.

The experience of re-reading the note has been a great reminder of the disconnect between what we think we're doing, and what we actually are doing! It shows the importance of writing out, or publically presenting, or in any case being explicit about our ideas. Only by doing so can we find the cracks and holes in our logic that our subconscious fills in automatically. (And with god-knows-what.)

NYC, 3 November 2009

Jeremy Weissmann
jeremy@mathmeth.com