Exercise 1 from WF122

Our next exercise is to design a proof of:

$$(0) \qquad [\ X \wedge (X \equiv Y) \ \equiv\ X \wedge Y \ ] \qquad .$$

Demonstrandum (0) is an equivalence; each equivaland is a conjunction, and each has X as a conjunct. The other conjunct contains Y as a subexpression in each case. Given the syntactic similarity between the equivalands, it is sweetly reasonable to aim for:

$$
\begin{array}{lll}
 & & X \wedge (X \equiv Y) \\
(*) & \equiv & \quad \{ \ldots \} \\
 & & X \wedge Y
\end{array}
$$

or its reverse as a proof shape.

Before investigating this approach, however, I want to do something a little more "learned" , based on the shape of (0) . I appeal to the "contextualization" formula:

$$(1) \qquad [\ A \Rightarrow (P \equiv Q) \ \equiv\ A \wedge P \ \equiv\ A \wedge Q \ ]$$

from the predicate calculus, to rewrite (0) equivalently as:

$$(0') \qquad [\ X \Rightarrow (\,(X \equiv Y) \equiv Y\,)\ ] \qquad .$$

We can calculate (0') in many ways. For example, the shape of (0') might suggest a contextual calculation:

⟦ (Punctual) context: X

$$
\begin{array}{ll}
 & X \equiv Y \\
\equiv & \quad \{ \text{ context } \} \\
 & \mathbf{true} \equiv Y \\
\equiv & \quad \{ \ \mathbf{true} \text{ is the unit of } \equiv \ \} \\
 & Y
\end{array}
$$

⟧ ,

or even:

$$
\begin{array}{ll}
 & (X \equiv Y) \equiv Y \\
\equiv & \quad \{ \text{ associativity of } \equiv \ \} \\
 & X \equiv (Y \equiv Y) \\
\equiv & \quad \{ \ Y \equiv Y \ \text{ is the unit of } \equiv \ \} \\
 & X \qquad ,
\end{array}
$$

which proves the stronger:

$$[\ X \ \equiv \ ((X \equiv Y) \equiv Y) \ ] \qquad ,$$

and so we see that formulae (0) and (0') are just consequences of the symmetry and associativity of $\equiv$.

So the above mainly goes to show the great utility of context, and formulae like (1) .

Let us return now to our earlier effort, with proof shape $(*)$. I opt for the forward direction of $(*)$, which amounts to eliminating X and $\equiv$ from the subexpression $X \equiv Y$, and also to conjoining X and Y .

As to the former goal, we want to eliminate $\equiv$ via its unit **true** , and thus we might eliminate X by rewriting it as **true** . To do this we need $X \, (\equiv \mathbf{true})$ in the context —punctual context is okay, because $\equiv$ is punctual— and indeed we do: that is, we have:

$$[\ X \ \Rightarrow \ (f.X \ \equiv \ f.\mathbf{true}) \ ] \qquad \text{or}$$

$$[\ X \wedge f.X \ \ \equiv \ \ X \wedge f.\mathbf{true} \ ]$$

for punctual f , and so we may calculate:

$$
\begin{array}{ll}
 & X \wedge (X \equiv Y) \\
\equiv & \quad \{ \text{ punctuality of } \equiv \ \} \\
 & X \wedge (\mathbf{true} \equiv Y) \\
\equiv & \quad \{ \ \mathbf{true} \ \text{ is the unit of } \equiv \ \} \\
 & X \wedge Y \qquad ,
\end{array}
$$

which establishes $(*)$ and hence (0) .

If we focus instead on the latter goal of conjoining X and Y , distributivity is suggested:

$$\begin{array}{ll} & X \wedge (X \equiv Y) \\ \equiv & \{ \wedge \text{ almost over } \equiv \} \\ & X \wedge X \equiv X \wedge Y \equiv X \\ \equiv & \{ \text{idempotence of } \wedge \text{, identity of } \equiv \} \\ & X \wedge Y \quad . \end{array}$$

So both reasonable considerations yield the same result.

* * *

I can't think of anything else to add about (0) . These experiments are proving quite pleasant! (Especially with a better pen!)

"Tynan's" , 10 September 2009

Commentary: What a difference a day makes. This note was written the day after EX0 , but the difference in clarity of exposition is considerable. When you turn on a hose that hasn't been used in awhile, the first burst of water is a little rusty. Clearly I didn't have too much buildup in my pipes! (Okay, maybe this analogy is a little gross.)

NYC, 4 November 2009

Jeremy Weissmann
jeremy@mathmeth.com