After a hiatus

It has been many months since my first EXs were written. The experiment embodied by those exercises was a great success — I really felt I harvested, gathered my mental energies, which had after the very emotional 2009 begun to unfocus.

For no particular reason —though undoubtedly relating in some way to my recent surge of creativity and activity in general— , I decided to do a few more exercises. Here we go!

My first exercise is to derive a calculational proof of:

(0) $[\ \mathbf{true} \Rightarrow X \ \equiv\ X\ ]$.

As usual, we should begin with a close analysis of the structure of this formula. It is an equivalence with equivalands $\mathbf{true} \Rightarrow X$ and X . The unknown structure X is conspicuously present in both, which on the surface suggests a (narrow) calculation beginning with $\mathbf{true} \Rightarrow X$ and ending with X , with equivalences at each step. (That suggestion would likely not have come to mind had the expressions been any more complex.)

Now we need to learn a bit about our ingredients, namely **true** and $\Rightarrow$. We shall take as given about **true** only:

(1) $[\ \mathbf{true} \equiv X \equiv X\ ]$,

which is, I think, restrictive enough to make this investigation interesting!

Ah no: I spoke too soon. In Dijkstra and Scholten's monograph "Predicate Calculus and Program Semantics" , the operator $\Rightarrow$ is defined in terms of $\equiv$ and $\vee$ both, so presumably we also need a property relating **true** and $\vee$, say:

(2) $[\ \mathbf{true} \vee X \ \equiv\ \mathbf{true}\ ]$.

We shall see!

We certainly need a property of $\Rightarrow$, and so I propose to use:

(3) $[\ X \Rightarrow Y \ \equiv\ X \vee Y \ \equiv\ Y\ ]$.

Thus the first step of our main calculation is:

$$
\begin{array}{ll}
 & \textbf{true} \Rightarrow X \\
\equiv & \quad \{ \ (3) \ \} \\
 & \textbf{true} \vee X \ \equiv \ X \qquad .
\end{array}
$$

If we possess both (1) and (2), we can continue:

$$
\begin{array}{ll}
 & \textbf{true} \vee X \ \equiv \ X \\
\equiv & \quad \{ \ (2) \ \} \\
 & \textbf{true} \ \equiv \ X \\
\equiv & \quad \{ \ (1) \ \} \\
 & X \qquad .
\end{array}
$$

But what if we only have (1) ? Let us try:

$$
\begin{array}{ll}
 & \textbf{true} \vee X \ \equiv \ X \\
\equiv & \quad \{ \ (1) \text{, to eliminate } \textbf{true} \ \} \\
 & (X \equiv X) \vee X \ \equiv \ X \\
\equiv & \quad \{ \ \vee \text{ over } \equiv \text{, to homogenize } \} \\
 & X \vee X \ \equiv \ X \vee X \ \equiv \ X \\
\equiv & \quad \{ \ (1) \ \} \\
 & \textbf{true} \ \equiv \ X \\
\equiv & \quad \{ \ (1) \ \} \\
 & X \qquad .
\end{array}
$$

And so we see that at the cost of " $\vee$ over $\equiv$ ", a property connecting $\vee$ and $\equiv$, we can do without property (2) !

Is this because we can prove (2) ? Let's see:

$$
\begin{array}{ll}
 & \mathbf{true} \vee X \\
\equiv & \{ \ (1) \ \} \\
 & (Y \equiv Y) \vee X \\
\equiv & \{ \ \vee \text{ over } \equiv \ \} \\
 & Y \vee X \ \equiv \ Y \vee X \\
\equiv & \{ \ (1) \ \} \\
 & \mathbf{true} \qquad .
\end{array}
$$

And there we have a proof of (2) .

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Postscript

I'm not pleased with my lazy syntactic analysis and the corresponding design of a proof shape. I blame my pen. (Or paper? or lack of writing surface?)

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