Exercise 2 from WF122

Next comes the intriguing formula:

(0) $[\ (X \equiv X \wedge Y) \ \vee \ (Y \equiv X \wedge Y)\ ]$.

We observe that (0) is symmetric in X and Y .

What sort of possibilities for manipulation does (0) provide? Because we do not know how to present a disjunction in a narrow calculation, we have to start wide. (We could use $[\ A \vee B \ \equiv\ \neg A \Rightarrow B\ ]$ to transform (0) . I'll explore this decidedly complificating approach last.)

One approach that obviously suggests itself is to distribute $\vee$ over $\equiv$. Another that comes to mind is that the shape $P \equiv P \wedge Q$ can be rewritten as $P \Rightarrow Q$ or $\neg P \vee Q$, and this last is very obviously compatible with $\vee$ in (0) !

So let's explore this last possibility first:

$(X \equiv X \wedge Y) \ \vee \ (Y \equiv X \wedge Y)$

$\equiv \quad \{ \Rightarrow \}$

$(X \Rightarrow Y) \ \vee \ (Y \Rightarrow X)$

$\equiv \quad \{ \Rightarrow \}$

$\neg X \vee Y \vee \neg Y \vee X$

$\equiv \quad \{\ [\ P \vee \neg P \ \equiv\ \mathbf{true}\ ]\ \}$

$\mathbf{true}$.

Easy. Now, we can expect distributing $\vee$ over $\equiv$ to take longer and to be more complicated, but let's try it for exploration's sake:

$$(X \equiv X \wedge Y) \;\vee\; (Y \equiv X \wedge Y)$$
$\equiv$ { $\vee$ over $\equiv$ }
$$X \vee Y \;\equiv\; X \vee (X \wedge Y) \;\equiv\; Y \vee (X \wedge Y) \;\equiv\; (X \wedge Y) \vee (X \wedge Y)$$
$\equiv$ { absorption on the middle two terms; idempotence on the last }
$$X \vee Y \;\equiv\; X \;\equiv\; Y \;\equiv\; X \wedge Y$$
$\equiv$ { Golden Rule }

true .

It was worth it, firstly for the exercise of a "messy" distribution, and secondly, to see the Golden Rule magically appear!

* * *

Finally, let's explore what may be a disaster, namely, turning (0) into an implication so that a narrow calculation might be possible. We begin with a wide manipulation of the body of (0) :

$$(X \equiv X \wedge Y) \;\vee\; (Y \equiv X \wedge Y)$$
$\equiv$ { $\Rightarrow$ }
$$\neg(X \equiv X \wedge Y) \;\Rightarrow\; (Y \equiv X \wedge Y) \quad ,$$

and then ask how we might establish:

(1) $[\; \neg(X \equiv X \wedge Y) \;\Rightarrow\; (Y \equiv X \wedge Y) \;]$.

I see two ways to proceed: In one approach, we would weaken $\neg(X \equiv X \wedge Y)$ into $Y \equiv X \wedge Y$, or strengthen $Y \equiv X \wedge Y$ into $\neg(X \equiv X \wedge Y)$. In another, we place $\neg(X \equiv X \wedge Y)$ in the context and calculate $Y \equiv X \wedge Y$; again, this last we might do widely —by calculating with all of $Y \equiv X \wedge Y$ — or narrowly —by manipulating one of Y and $X \wedge Y$ into the other— .

So many possibilities, and all seem fun! Nothing to do but try them all.

No Context

I opt to weaken $\neg(X \equiv X \wedge Y)$ into $Y \equiv X \wedge Y$, as the former expression affords more manipulative possibilities. Also, I know I must weaken somewhere, as equivalence does not hold. The most obvious weakening I can think of is " $\not\equiv$ implies $\vee$ " . It's worth a try:

$\neg(X \equiv X \wedge Y)$

$\equiv$ { rewriting }

$X \not\equiv X \wedge Y$

$\Rightarrow$ { $\not\equiv$ implies $\vee$ }

$X \vee (X \wedge Y)$

$\equiv$ { absorption }

X .

Now, at this point I could easily follow up with either:

$[\![$ Context: X —which follows from $\neg(X \equiv X \wedge Y)$ as above—

$Y \equiv X \wedge Y$

$\equiv$ { context }

$Y \equiv \textbf{true} \wedge Y$

$\equiv$ { predicate calculus }

true

$]\!]$

or:

$[\![$ Context: X

$X \wedge Y$

$\equiv$ { context }

Y

$]\!]$.

But what if I wanted to continue the original non-contextual calculation? Let's try an experiment:

$\neg(X \equiv X \wedge Y)$

$\Rightarrow$ { as above }

X

$\equiv$ { Aiming for $Y \equiv X \wedge Y$, so I need two Y's , $\equiv$, and $\wedge$. I can think of one possibility, but this is just an experiment. }

$X \wedge \mathbf{true}$

$\equiv$ { ... }

$X \wedge (Y \equiv Y)$

$\equiv$ { Now I need X and Y as conjuncts, so maybe try distributivity? }

$X \wedge Y \equiv X \wedge Y \equiv X$

$\equiv$ { The problem now is that I need Y by itself, and I don't know how to do this. I can reuse the first line... }

$X \wedge Y \equiv \mathbf{false}$.

That didn't really work. Perhaps the best we can do is:

X

$\equiv$ { predicate calculus }

$X \equiv \mathbf{true}$

$\equiv$ { punctual Leibniz }

$X \wedge Y \equiv \mathbf{true} \wedge Y$

$\equiv$ { predicate calculus }

$X \wedge Y \equiv Y$,

which is just a wide presentation of the above contextual calculations. Aha! We should have opted to invert the calculation at this point, as X no longer affords nice manipulative possibilities:

$X \wedge Y \ \equiv \ Y$

$\equiv$ { predicate calculus, aiming to eliminate Y via Leibniz }

$X \wedge Y \ \equiv \ \mathbf{true} \wedge Y$

$\Leftarrow$ { Leibniz }

$X \equiv \mathbf{true}$

$\equiv$ { predicate calculus }

X .

I think that's a nice context-free calculation of (1) , though it does force us to work from both sides.

With Context

So now I put $\neg(X \equiv X \wedge Y)$ in the context and aim to calculate $Y \equiv X \wedge Y$. I do this by aiming to massage $X \wedge Y$ into Y via equivalence-preserving manipulations:

$[\![$ Context: $\neg(X \equiv X \wedge Y)$

$X \wedge Y$

$\equiv$ { I could rewrite the whole expression as $\neg X$ using the context, but that would remove my manipulative possibilities, along with the target Y . }

$\neg(X \wedge Y) \wedge Y$

$\equiv$ { de Morgan }

$(\neg X \vee \neg Y) \wedge Y$

$\equiv$ { using right conjunct to rewrite the left conjunct }

$(\neg X \vee \mathbf{false}) \wedge Y$

$\equiv$ { predicate calculus }

$\neg X \wedge Y$

$]\!]$.

EX2–5

I realize after calculating blindly that our context could not help us, as it does not cater to the removal/introduction of X ! So perhaps this sort of narrow calculation will not work.

But wait! The result of this calculation is:

$X \wedge Y \equiv \neg X \wedge Y$

$\equiv$ { factoring out Y }

$Y \Rightarrow (X \equiv \neg X)$

$\equiv$ { $X \equiv \neg X \equiv$ **false** , predicate calculus }

$\neg Y$.

So we have:

⟦ Context: $\neg Y$ —which follows from $\neg(X \equiv X \wedge Y)$ as above—

$Y \equiv X \wedge Y$

$\equiv$ { context }

false $\equiv X \wedge$ **false**

$\equiv$ { predicate calculus }

true

⟧ .

What intrigues me about the above is that we showed separately that both X and $\neg Y$ follow from $\neg(X \equiv X \wedge Y)$. Oh, but of course!

$\neg(X \equiv X \wedge Y)$

$\equiv$ { implication }

$\neg(X \Rightarrow Y)$

$\equiv$ { predicate calculus }

$X \wedge \neg Y$.

So in fact, our context *equivales* $X \wedge \neg Y$! But then to prove $Y \equiv X \wedge Y$, we only need $X \vee \neg Y$; indeed:

$$Y \equiv X \wedge Y$$

$\equiv$ { implication }

$$Y \Rightarrow X$$

$\equiv$ { implication }

$$\neg Y \vee X \quad .$$

So now we have the full picture of (1) :

$$\neg(X \equiv X \wedge Y) \quad \Rightarrow \quad (Y \equiv X \wedge Y)$$

$\equiv$ { decoding }

$$X \wedge \neg Y \quad \Rightarrow \quad X \vee \neg Y \quad .$$

This picture can be generalized naturally:

$$\langle \forall X :: X \rangle \quad \Rightarrow \quad \langle \exists X :: X \rangle \qquad \text{(for nonempty range)}$$

or, to phrase it more like (0) :

$$\langle \exists X :: \neg X \rangle \quad \vee \quad \langle \exists X :: X \rangle \quad .$$

In fact, the proper generalization of (0) would be:

$$[\ \ \langle \exists X \ :: \ X \equiv \langle \forall X :: X \rangle \ \rangle \ \ ] \quad .$$

This all vaguely smells of some of my JAWs on skolemization. To be continued.

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