Exercise 3 from WF122

Today's exercise is:

(0) $[\ X \wedge (X \Rightarrow Y) \ \equiv \ X \wedge Y \ ]$,

which is the equivalence version of "Modus Ponens" :

(1) $[\ X \wedge (X \Rightarrow Y) \ \Rightarrow \ Y \ ]$.

Formula (0) gives "the whole story" behind the familiar inference rule (1) .

I think it is worth comparing (0) to:

(2) $[\ X \wedge (X \equiv Y) \ \equiv \ X \wedge Y \ ]$,

which was the subject matter of EX1 . Because $X \wedge (X \equiv Y)$ is stronger than $X \wedge (X \Rightarrow Y)$, we immediately conclude:

(0a) $[\ X \wedge (X \Rightarrow Y) \ \Leftarrow \ X \wedge Y \ ]$.

provided we wish to use the results of our previous investigation. It then remains to prove:

(0b) $[\ X \wedge (X \Rightarrow Y) \ \Rightarrow \ X \wedge Y \ ]$.

However, it is not clear to me that this approach buys us anything, as (0b) does not seem any more straightforward than the original demonstrandum (0) . But let us press on, for completeness's sake. I would split (0b) further into the conjunction of:

(0c) $[\ X \wedge (X \Rightarrow Y) \ \Rightarrow \ X \ ]$ and

(1) $[\ X \wedge (X \Rightarrow Y) \ \Rightarrow \ Y \ ]$.

Formula (0c) is just weakening, while (1) may be calculated as follows:

$[\![$ Context: $X \wedge (X \Rightarrow Y)$

Y

$\Leftarrow$ { context: $X \Rightarrow Y$ }

X

$\equiv$ { context: X }

true

$]\!]$.

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If one revisits EX1 , one will find that in fact most of the heuristic arguments and calculations hold equally well here. Hence, rather than repeat those arguments, we ought to generalize them.

The shape of (0) and (2) is:

(3) $[\ X \wedge f.X.Y \ \equiv \ X \wedge Y\ ]$.

To calculate (3) , we follow a strategy from EX1 which worked well there. First we massage:

$X \wedge f.X.Y \ \equiv \ X \wedge Y$

$\equiv$ { predicate calculus }

$X \Rightarrow (f.X.Y \equiv Y)$

and then follow up with a contextual calculation:

$[\![$ Context: X

$f.X.Y$

$\equiv$ { • f is punctual in its left argument }

$f.\mathbf{true}.Y$

$\equiv$ { • $[\ f.\mathbf{true}.Y \equiv Y\ ]$ }

Y

$]\!]$.

Thus we see that two sweetly reasonable conditions on f so that (3) follows are:

- f is punctual in its left argument
- $[\ f.\mathbf{true}.Y \equiv Y\ ]$.

Introducing binary boolean operator $\rightarrow$ and writing $[\ f.X.Y \equiv X \rightarrow Y\ ]$, we might write these conditions as:

- $\rightarrow$ is punctual (in its left argument)
- **true** is a (left) identity for $\rightarrow$.

Clearly, then, we may take any of $\wedge$, $\Rightarrow$, or $\equiv$ for $\rightarrow$, whence all of:

$$[\ X \wedge (X \wedge Y) \ \equiv\ X \wedge Y\ ]$$

$$[\ X \wedge (X \Rightarrow Y) \ \equiv\ X \wedge Y\ ]$$

$$[\ X \wedge (X \equiv Y) \ \equiv\ X \wedge Y\ ]$$

follow from (3) .

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While writing up this exercise, I found myself asking if any of this was interesting or useful. That reminded me of when my 8th grade math students would ask: “When are we going to need to know any of this?” . My answer was always: “What do you mean by ‘this’ ? Do you mean ‘the distributive property’ ? Because then the answer is: ‘Probably never.’ . But if by ‘this’ you mean the skill to focus on a kind of problem and learn to solve it skillfully and gracefully, then the answer is: ‘Every day, for the rest of your lives.’ .” . (Probably I never said it exactly like that!)

The point is that this exercise was worth it, even if all I did was practice my handwriting. Or barring that, then at least it was time to focus on *something* , no matter how trivial or useless.

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