Exercise 4 from WF122

Today's exercise is the classic shunting formula:

$$(0) \qquad [\ X \wedge Y \ \Rightarrow\ Z \quad \equiv \quad X \Rightarrow (Y \Rightarrow Z)\ ] \qquad .$$

Experienced calculators use (0) all the time to move conjuncts between the context and the antecedent of an implication. In fact, shunting is so ubiquitous that I have no idea whether or not I used it in previous EX's !

Before beginning, I wish to note that shunting embodies a kind of associativity, in that it achieves syntactic regrouping of the middle term Y . It differs from usual associativity in that there is a semantic change from $\wedge$ to $\Rightarrow$. Compare (0) with the familiar arithmetic formula:

$$(1) \qquad (x^y)^z \ = \ x^{yz} \qquad .$$

It is hard to see the similarity because of all the invisible symbols. But if I write $\nearrow$ for exponentiation and $*$ for multiplication, (1) becomes:

$$(1') \qquad (x \nearrow y) \nearrow z \quad = \quad x \nearrow (y * z) \qquad ,$$

and the similarity becomes clear. The beauty of this sort of associativity is also discussed in JAW101 .

Now that we have drawn attention to the associative nature of (0) , it is clear that our proof of (0) has to involve the syntactic regrouping of Y . I envision a narrow proof shape, either:

$$\begin{array}{ll} & X \wedge Y \ \Rightarrow\ Z \\ \equiv & \quad \{ \ ... \ \} \\ & X \Rightarrow (Y \Rightarrow Z) \end{array}$$

or the reverse. I cannot see an obvious way to distinguish the two, so I defer the choice.

Knowing that we need associativity, clearly we need to eliminate the unassociative $\Rightarrow$ in favor of an associative operator like $\equiv$, $\wedge$, or $\vee$. Since we are rewriting $\Rightarrow$, I prefer to begin my manipulations with $X \wedge Y \ \Rightarrow\ Z$, which has only one occurrence of $\Rightarrow$. Also, the presence of $\wedge$ suggests we rewrite $\Rightarrow$ using:

(2) $[\ P \Rightarrow Q \ \equiv \ P \ \equiv \ P \wedge Q\ ]$.

Thus we begin:

$X \wedge Y \ \Rightarrow \ Z$

$\equiv$ { (2) with $P,Q \ := \ X \wedge Y\,,\, Z$ }

$X \wedge Y \ \equiv \ (X \wedge Y) \wedge Z$

$\equiv$ { associativity of $\wedge$ }

$X \wedge Y \ \equiv \ X \wedge (Y \wedge Z)$.

So far so good! We accomplished the regrouping of Y by forming expression $Y \wedge Z$. Noting that our goal contains subexpression $Y \Rightarrow Z$, it is sweetly reasonable to try to use (2) again to form this subexpression. But to use (2) , we need Y and $Y \wedge Z$ to be linked by $\equiv$, whereas currently they are both *contained in* expressions linked by $\equiv$. This is a distributivity shape! This conclusion is further bolstered by the fact that both Y and $Y \wedge Z$ are conjoined with the same expression X . Since we know how to distribute $\wedge$ over $\equiv$:

(3) $[\ P \wedge (Q \equiv R) \ \equiv \ P \wedge Q \ \equiv \ P \wedge R \ \equiv \ P\ ]$,

we may continue:

$X \wedge Y \ \equiv \ X \wedge (Y \wedge Z)$

$\equiv$ { (3) with $P\,,\,Q\,,\,R \ := \ X\,,\,Y\,,\,Y \wedge Z$ }

$X \ \equiv \ X \wedge (Y \equiv Y \wedge Z)$

$\equiv$ { (2) with $P,Q \ := \ Y,Z$ }

$X \ \equiv \ X \wedge (Y \Rightarrow Z)$

$\equiv$ { (2) with $P,Q \ := \ X\,,\,Y \Rightarrow Z$ }

$X \ \Rightarrow \ (Y \Rightarrow Z)$.

Viola!

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This is by far the nicest design of a proof I have constructed in a long time. What a wonderful exercise!

Chokolat, 14 September 2009

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As a sort of postlude, I record another proof of (0) , using:

(4) $[\ P \Rightarrow Q \quad \equiv \quad \neg P \vee Q\ ]$.

The design is less straightforward, in my opinion:

$X \wedge Y \ \Rightarrow\ Z$

$\equiv$ { Using (4) with $P, Q := X \wedge Y, Z$ changes $\Rightarrow$ to $\vee$, and negates the conjunction. From de Morgan we know that the negation of a conjunction is a disjunction, so we will eventually be able to rewrite the whole expression in terms of the associative $\vee$. }

$\neg(X \wedge Y) \ \vee\ Z$

$\equiv$ { de Morgan }

$(\neg X \ \vee\ \neg Y) \ \vee\ Z$

$\equiv$ { associativity of $\vee$ }

$\neg X \ \vee\ (\neg Y \ \vee\ Z)$

$\equiv$ { (4) with $P, Q := Y, Z$ }

$\neg X \ \vee\ (Y \Rightarrow Z)$

$\equiv$ { (4) with $P, Q := X, Y \Rightarrow Z$ }

$X \ \Rightarrow\ (Y \Rightarrow Z)$.

This proof is nice, but the first step takes a lot of looking ahead.

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