Distributivity of implication

In EX5 , the following distributivity formula was proved:

$$[\quad X \Rightarrow (Y \Rightarrow Z) \quad \equiv \quad (X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \quad] \qquad ;$$

in other words, " $\Rightarrow$ distributes over $\Rightarrow$ from the left" , or perhaps more accurately, " $(X\Rightarrow)$ distributes over $\Rightarrow$ " .

In this note, I would like to explore distributivity from the right, in general:

(0) $\quad [\quad (X \diamond Y) \Rightarrow Z \quad \equiv \quad ??? \quad] \qquad .$

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When it comes to distributivity, $\vee$ is certainly the nicest operator, as it distributes over $\equiv$ and $\vee$ by postulate, and hence over $\wedge$ and $\Rightarrow$ as well, which are defined in terms of $\equiv$ and $\vee$.

So I propose to begin our exploration as follows:

$(X \diamond Y) \Rightarrow Z$

$\equiv \quad \{ \Rightarrow$ into $\neg/\vee \}$

$\neg(X \diamond Y) \vee Z$

$\equiv \quad \{$ see below: $\neg$ over $\diamond \}$

$(f.X \circ g.Y) \vee Z$

$\equiv \quad \{$ see below: $\vee$ over $\circ \}$

$(f.X \vee Z) \circ (g.Y \vee Z)$

$\equiv \quad \{ \neg/\vee$ into $\Rightarrow \}$

$(\neg f.X \Rightarrow Z) \circ (\neg g.Y \Rightarrow Z) \qquad .$

This essentially completes the investigation. We just need to refine $\diamond$, $\circ$, f , and g , according to:

(1a) $\quad [\quad \neg(X \diamond Y) \quad \equiv \quad f.X \circ g.Y \quad]$

(1b) $\quad \circ$ is among $\wedge$, $\vee$, $\equiv$, $\Rightarrow$, $\Leftarrow \qquad .$

If we want to refine (0) into a "pure" distributivity shape, we might want to take $[\ f.X \equiv \neg X\ ]$ and $[\ g.X \equiv \neg X\ ]$, so that (0) becomes:

$$[\quad (X \diamond Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \circ (Y \Rightarrow Z) \quad]$$

on account of double negation. Then (1a) becomes:

$$[\quad \neg(X \diamond Y) \quad \equiv \quad \neg X \circ \neg Y \quad] \qquad ,$$

which has the obvious instantiations $\diamond , \circ := \wedge , \vee$ and $\diamond , \circ := \vee , \wedge$, and the less obvious $\diamond , \circ := \not\equiv , \equiv$. (Also $\diamond , \circ := \equiv , \not\equiv$ would work, but then (1b) is not satisfied: note that $\vee$ does not distribute over $\not\equiv$!) So we have:

$$[\quad X \wedge Y \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \vee (Y \Rightarrow Z) \quad]$$

$$[\quad X \vee Y \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \wedge (Y \Rightarrow Z) \quad]$$

$$[\quad (X \not\equiv Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \equiv (Y \Rightarrow Z) \quad] \qquad .$$

Nice!

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This choice for f and g settles the choices of $\wedge$, $\vee$, and $\not\equiv$ for $\diamond$. So now let us investigate $\Rightarrow$, $\Leftarrow$, and $\equiv$ for $\diamond$.

If we take $\diamond := \Rightarrow$, then (1a) becomes:

$$[\quad \neg(X \Rightarrow Y) \quad \equiv \quad f.X \circ g.Y \quad] \qquad ,$$

so the choices $[\ f.X \equiv X\ ]$ and $[\ g.Y \equiv \neg Y\ ]$ and $\circ = \wedge$ are forced upon us, yielding the following refinement of (0) :

$$[\quad (X \Rightarrow Y) \Rightarrow Z \quad \equiv \quad (\neg X \Rightarrow Z) \wedge (Y \Rightarrow Z) \quad] \qquad \text{or}$$

$$[\quad (X \Rightarrow Y) \Rightarrow Z \quad \equiv \quad (X \vee Z) \wedge (Y \Rightarrow Z) \quad] \qquad .$$

Similarly, we get:

$$[\quad (X \Leftarrow Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \wedge (\neg Y \Rightarrow Z) \quad] \qquad \text{or}$$

$$[\quad (X \Leftarrow Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \wedge (Y \vee Z) \quad] \qquad .$$

Finally, if we take $\diamond := \equiv$, then (1a) becomes:

$$[\quad \neg(X \equiv Y) \quad \equiv \quad f.X \circ g.Y \quad] \qquad .$$

On account of:

$$[\quad \neg(X \equiv Y) \quad \equiv \quad \neg X \equiv Y \quad] \qquad \text{and}$$

$$[\quad \neg(X \equiv Y) \quad \equiv \quad X \equiv \neg Y \quad]$$

we take $\circ := \equiv$, and either:

$$[\ f.X \equiv \neg X\ ] \qquad \text{and} \qquad [\ g.X \equiv X\ ] \qquad , \qquad \text{or}$$

$$[\ f.X \equiv X\ ] \qquad \text{and} \qquad [\ g.X \equiv \neg X\ ] \qquad .$$

The former choice yields:

$$[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \Rightarrow Z \quad \equiv \quad \neg Y \Rightarrow Z \quad] \qquad \text{or}$$

$$[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \Rightarrow Z \quad \equiv \quad Y \vee Z \quad]$$

while the latter choice yields:

$$[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad \neg X \Rightarrow Z \quad \equiv \quad Y \Rightarrow Z \quad] \qquad \text{or}$$

$$[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \vee Z \quad \equiv \quad Y \Rightarrow Z \quad] \qquad .$$

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In sum, we have the following refinements of (0) :

- $[\quad X \wedge Y \ \Rightarrow\ Z \quad \equiv \quad (X \Rightarrow Z) \vee (Y \Rightarrow Z) \quad]$

- $[\quad X \vee Y \ \Rightarrow\ Z \quad \equiv \quad (X \Rightarrow Z) \wedge (Y \Rightarrow Z) \quad]$

- $[\quad (X \not\equiv Y) \Rightarrow Z \quad \equiv \quad X \Rightarrow Z \quad \equiv \quad Y \Rightarrow Z \quad]$

- $[\quad (X \Rightarrow Y) \Rightarrow Z \quad \equiv \quad (\neg X \Rightarrow Z) \ \wedge\ (Y \Rightarrow Z) \quad]$

  $[\quad (X \Rightarrow Y) \Rightarrow Z \quad \equiv \quad (X \vee Z) \ \wedge\ (Y \Rightarrow Z) \quad]$

- $[\quad (X \Leftarrow Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \ \wedge\ (\neg Y \Rightarrow Z) \quad]$

  $[\quad (X \Leftarrow Y) \Rightarrow Z \quad \equiv \quad (X \Rightarrow Z) \ \wedge\ (Y \vee Z) \quad]$

- $[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \Rightarrow Z \quad \equiv \quad \neg Y \Rightarrow Z \quad]$

  $[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \Rightarrow Z \quad \equiv \quad Y \vee Z \quad]$

  $[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad \neg X \Rightarrow Z \quad \equiv \quad Y \Rightarrow Z \quad]$

  $[\quad (X \equiv Y) \Rightarrow Z \quad \equiv \quad X \vee Z \quad \equiv \quad Y \Rightarrow Z \quad]$.

What a lovely investigation!

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