Exercise 6 from WF122

Our next exercise —after a bit of a hiatus— is:

(0) $[\ (X \Rightarrow Y) \vee (Y \Rightarrow Z)\ ]$.

How does one prove a disjunction? This is always a difficulty. Some approaches are:

- manipulate the disjunction into something else
- use $[\ P \vee Q \ \equiv\ \neg P \Rightarrow Q\ ]$, contextualize $\neg P$, and calculate Q
- embark on a case analysis .

As for the first approach, we are relatively unguided: how to manipulate (0) ? Because $[\ P \vee Q \ \equiv\ \neg P \Rightarrow Q\ ]$ is our second approach, let's not manipulate the disjunction as a whole, but rather some of its subexpressions, namely $X \Rightarrow Y$ and $Y \Rightarrow Z$. And how to manipulate these? In the interest of homogeneity, we might turn them into disjunctions. To my mind this leaves a simple possibility:

$$
\begin{array}{ll}
 & (X \Rightarrow Y) \vee (Y \Rightarrow Z) \\
\equiv & \{ \ \Rightarrow \text{ into } \neg/\vee \ \} \\
 & (\neg X \vee Y) \vee (\neg Y \vee Z) \\
\equiv & \{ \text{ associativity of } \vee \ \} \\
 & \neg X \vee (Y \vee \neg Y) \vee Z \\
\equiv & \{ \text{ predicate calculus } \} \\
 & \neg X \vee \mathbf{true} \vee Z \\
\equiv & \{ \text{ predicate calculus } \} \\
 & \mathbf{true} \quad ,
\end{array}
$$

and a complex possibility:

EX6–1

$$(X \Rightarrow Y) \;\vee\; (Y \Rightarrow Z)$$

$\equiv$ { $\Rightarrow$ into $\equiv / \vee$ }

$$(Y \equiv X \vee Y) \quad \vee \quad (Z \equiv Y \vee Z)$$

$\equiv$ { $\vee$ over $\equiv$ }

$$Y \vee Z \;\equiv\; Y \vee (Y \vee Z) \;\equiv\; (X \vee Y) \vee Z \;\equiv\; (X \vee Y) \vee (Y \vee Z)$$

$\equiv$ { associativity and idempotence of $\vee$ }

$$Y \vee Z \;\equiv\; Y \vee Z \;\equiv\; X \vee Y \vee Z \;\equiv\; X \vee Y \vee Z$$

$\equiv$ { predicate calculus }

$$\mathbf{true} \equiv \mathbf{true}$$

$\equiv$ { predicate calculus }

$$\mathbf{true} \quad .$$

I was quite happy to see the latter work out!

As for the second possibility, (0) becomes either:

$$[\; \neg(X \Rightarrow Y) \;\Rightarrow\; (Y \Rightarrow Z) \;] \qquad \text{or}$$

$$[\; \neg(Y \Rightarrow Z) \;\Rightarrow\; (X \Rightarrow Y) \;] \qquad .$$

Using $[\, \neg(P \Rightarrow Q) \equiv P \wedge \neg Q \,]$, the two possibilities become:

$$[\; X \wedge \neg Y \;\Rightarrow\; (Y \Rightarrow Z) \;]$$

$$[\; Y \wedge \neg Z \;\Rightarrow\; (X \Rightarrow Y) \;] \qquad .$$

The contextual proofs are walkovers:

$[\![$ Context: $X \wedge \neg Y$

$$Y$$

$\equiv$ { context }

$$\mathbf{false}$$

$\equiv$ { predicate calculus }

$$Z$$

$]\!]$

and:

$[\![$ Context: $\quad Y \land \neg Z$

$\quad Y$

$\equiv \quad$ { context }

$\quad$ **true**

$\Leftarrow \quad$ { predicate calculus }

$\quad X$

$]\!]$.

Of course, we used here:

$$[\ \mathbf{false} \Rightarrow P\ ] \qquad \text{and} \qquad [\ P \Rightarrow \mathbf{true}\ ] \qquad ,$$

but these are also walkovers:

$\quad \mathbf{false} \Rightarrow P$

$\equiv \quad \{ \Rightarrow$ into $\neg / \lor \}$

$\quad \neg\, false \lor P$

$\equiv \quad$ { **true**/**false** }

$\quad \mathbf{true} \lor P$

$\equiv \quad$ { predicate calculus }

$\quad$ **true**

and:

$\quad P \Rightarrow \mathbf{true}$

$\equiv \quad \{ \Rightarrow$ into $\neg / \lor \}$

$\quad \neg P \lor \mathbf{true}$

$\equiv \quad$ { predicate calculus }

$\quad$ **true** .

EX6–3

Finally, we try the case analysis approach. To be honest, I'm not sure how to do this without simply repeating the second approach. So with that, I bring a close to this EX .

NYC, 14 October 2009

Jeremy Weissmann
jeremy@mathmeth.com