Exercise 7 from WF122

After another hiatus, we return with the classic formula:

$$(0) \qquad [\ \ (X \Rightarrow Y) \wedge (Y \Rightarrow X) \ \ \equiv \ \ X \ \ \equiv \ \ Y \ \ ] \qquad ,$$

otherwise known as "Mutual Implication" .

Since (0) is not an implication, there is no possibility of contextualization; thus we have fewer choices for how to begin — hooray!

Without being overly committal, I would like to explore manipulating the equivaland $(X \Rightarrow Y) \wedge (Y \Rightarrow X)$, which is to my mind the smallest interesting chunk of (0) . Perhaps we can manipulate it directly into $X \equiv Y$, completing the proof, and perhaps not. In either case our investigation should prove useful.

How to manipulate $(X \Rightarrow Y) \wedge (Y \Rightarrow X)$? We might wish to unfold something into an equivalence, since the other chunk of (0) is $X \equiv Y$. One possibility is to unfold $\wedge$ using the golden rule: I prefer this approach because it maintains the symmetry of the conjuncts and obviates the question of which implication to unfold:

$$
\begin{array}{ll}
 & (X \Rightarrow Y) \wedge (Y \Rightarrow X) \\
\equiv & \quad \{ \text{ Golden Rule } \} \\
 & X \Rightarrow Y \quad \equiv \quad Y \Rightarrow X \quad \equiv \quad (X \Rightarrow Y) \vee (Y \Rightarrow X) \\
\equiv & \quad \{ \text{ EX6 } \} \\
 & X \Rightarrow Y \quad \equiv \quad Y \Rightarrow X \qquad .
\end{array}
$$

Weird! But this is exciting: our implications are now joined by $\equiv$, and we know how to unfold $\Rightarrow$ into $\equiv$, using either:

$$[\ \ P \Rightarrow Q \ \ \equiv \ \ P \ \ \equiv \ \ P \wedge Q \ \ ] \qquad \text{or}$$

$$[\ \ P \Rightarrow Q \ \ \equiv \ \ Q \ \ \equiv \ \ Q \vee P \ \ ] \qquad .$$

Since we are aiming for $X \equiv Y$, it makes sense to use the same rewrite on both implications, so the resulting equivalence will have both X and Y as equivalands:

EX7–1

$$(X \Rightarrow Y) \equiv (Y \Rightarrow X)$$

$\equiv$ { $\Rightarrow$ into $\equiv/\wedge$ —arbitrary— }

$$(X \equiv X \wedge Y) \equiv (Y \equiv Y \wedge X)$$

$\equiv$ { predicate calculus —see below— }

$$X \equiv Y \quad ,$$

which magically completes the proof! (In the last step we used associativity and symmetry of $\equiv$, symmetry of $\wedge$, and properties of **true** .)

If we had used " $\Rightarrow$ into $\equiv/\vee$ " , the proof would of course have worked in the same way. What is perhaps a surprise is that if we had used both, the proof would still have worked out:

$$X \Rightarrow Y \equiv Y \Rightarrow X$$

$\equiv$ { $\Rightarrow$ into $\equiv/\wedge$, $\Rightarrow$ into $\equiv/\vee$ }

$$X \equiv X \wedge Y \equiv X \equiv X \vee Y$$

$\equiv$ { predicate calculus }

$$X \wedge Y \equiv X \vee Y$$

$\equiv$ { Golden Rule }

$$X \equiv Y \quad .$$

Of course, this is not much of a surprise: the Golden Rule is responsible for the equivalence of " $\Rightarrow$ into $\equiv/\wedge$ " and " $\Rightarrow$ into $\equiv/\vee$ " anyway.

Well, this was a delight!

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