An experiment with a geometry problem

The problem

Find the maximum area of an octagon such that:

(0) it is inscribed in a circle

(1) four alternating vertices of it induce a rectangle of area 4

(2) the other four vertices induce a square of area 5 .

We'll call such octagons **blorgs** , for brevity.

Notes on our approach

Our aim here is not only to solve this problem, but to design a solution in a disciplined fashion. Since we lack pleasant geometrical interfaces, we can expect in advance that the design will not be pleasant. Nevertheless, the experiment here is to do the best job we can, avoiding pictorial reasoning as much as possible. We can also expect that we will have to rely on intuition now and then, whenever we need to borrow some geometrical facts.

Enough whining: Let's begin!

Initial observations

A square of given area (here, 5) has fixed dimensions $(\sqrt{5}\times\sqrt{5})$, hence is inscribed in a circle of fixed radius $(\sqrt{10}/2)$ and fixed area $(\frac{5}{2}\cdot\pi)$. By (0) , the set of blorg areas is bounded above by this area, and hence has a least upper bound. However, it is not clear that this bound is attained by a blorg. Therefore, to truly solve the problem, we must not only find this bound, but also construct a blorg with that area: We are simultaneously investigating the maximal area, as well as the geometry of a "maximal blorg" .

Consider an arbitrary blorg. What can we say about its area? Naturally our eyes turn to (1) and (2) , which mention area, and thus we may ask: How do the areas of the rectangle and square relate to the area of the blorg? To answer this question, we have to see that, by (0) yet again, the induced rectangle and square are *inscribed* in the blorg. Hence the area of the blorg equals the area of one of these, plus a remainder.

A fork in the path?

So, we may view the blorg as a square plus a remainder, or a rectangle plus a remainder. At this point in our investigation, it would be tempting to relate the blorg to the square, since the geometry of the square is known, making it easier to draw an accurate picture. We will see later that this would be an unfortunate choice.

In fact, there is no need to make a choice! Since squares and rectangles are both rectangles, it would be wise to consider an arbitrary inscribed rectangle, and investigate the remainder.

Investigating the remainder

Now we are considering a fixed rectangle, and an arbitrary blorg which induces it with 4 alternate vertices. Since we have no way to distinguish the sides of the rectangle, we focus on one side, call him Peter.

Peter partitions the blorg into two "zones" , and since Peter's endpoints are alternate vertices of the blorg, this means that of the remaining 6 vertices not on Peter, 5 lie in one zone, and 1 —call her Mary— lies in the other. Since a rectangle has 4 vertices, the rectangle lies entirely in the former zone, and so the area induced by Peter and Mary is part of the remainder. Of course, this area is a triangle. So by symmetry, we may conclude that the entire remainder consists of 4 such triangles, each induced by a side of the rectangle (a Peter), and one unique vertex (his Mary).

Maximum sum, sum of maximums

Because the area of the rectangle is fixed, to calculate the maximal area of a blorg we have to calculate the maximum remainder, the maximum sum of the areas of the triangles. It is tempting to try to do so by calculating the maximum area of each triangle separately, but note that the maximum sum is **at most, but not necessarily equal to** the sum of the maximums! However, it is sweetly reasonable to investigate the latter, as it will give us an upper bound on the former. And moreover, if we can construct a blorg meeting this upper bound, we will have solved the problem.

The area of a triangle equals half the length of Peter, times the distance from Peter to Mary (Mary's altitude). Peter is the fixed side of a rectangle, so the maximum area of the triangle corresponds to the maximum altitude of Mary. With some basic geometry, it is not hard to see that this maximum altitude perpendicularly bisects Peter, and goes through the center of the circle. (See Appendices.)

Since the Peters intersect at right angles, so do the altitudes, implying that the Marys are spaced equidistantly around the circle — that is, they induce a square. And thus we have precisely determined the geometry of a maximal blorg: the diagonals of the square bisect the sides of the rectangle.

Computing the area

Now we turn to computing the area. Let the dimensions of the rectangle be x and y , and let r be the radius of the circle, so that we have:

(3) $\quad x \cdot y \ = \ 4$

(4) $\quad 2 \cdot r^2 \ = \ 5$

$\quad (x/2)^2 + (y/2)^2 \ = \ r^2 \qquad .$

Multiplying the last equation through by 4 , we get:

(5) $\quad x^2 + y^2 \ = \ 4 \cdot r^2 \qquad .$

The area of a triangle with Peter length x is:

$$
\begin{array}{ll}
 & \frac{1}{2} \cdot \text{Peter} \cdot \text{(Mary altitude)} \\
= & \quad \{ \quad \} \\
 & \frac{1}{2} \cdot x \cdot (r - \frac{y}{2}) \qquad ,
\end{array}
$$

and the area of a triangle with Peter length y is the same with x and y switched. With a little algebra and arithmetic, we thus find that the area we are looking for is the remarkable value:

$$(x + y) \cdot r \qquad .$$

Given (3)–(5) , it is easier to compute the square of this value, namely:

$$
\begin{array}{ll}
 & (x + y)^2 \cdot r^2 \\
= & \quad \{ \text{ algebra } \} \\
 & (x^2 + 2 \cdot x \cdot y + y^2) \cdot r^2 \\
= & \quad \{ \ (3) \text{ and } (5) \ \} \\
 & (4 \cdot r^2 + 8) \cdot r^2 \\
= & \quad \{ \ (4) \text{ , ie } r^2 = \frac{5}{2} \ \} \\
 & (10 + 8) \cdot \frac{5}{2} \\
= & \quad \{ \text{ arithmetic } \} \\
 & 45 \qquad .
\end{array}
$$

Hence the area of a maximal blorg is $\sqrt{45} = 3\cdot\sqrt{5}$, which is independent of x and y , the dimensions of the rectangle! This is quite interesting, because it is conceivable that, though the area of the rectangle is fixed, its Peters may vary, thus affecting the areas of the triangles. But in fact one can use properties (3)–(5) to calculate the dimensions of the rectangle: $2\cdot\sqrt{2}$ and $\sqrt{2}$. It turns out that the area and the dimensions of a rectangle inscribed in a circle determine each other, but I did not know this when I first attempted the problem.

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Afterword

I do feel that the funny names have helped the exposition quite a bit.

Like many of the lower-numbered JAWs , this one was conceived in late 2004 , but not written in full until more than two years later; in fact, this was the third or fourth proof I ever designed.

The last gap in the JAW series has now been filled!

Santa Cruz, 17 October 2006

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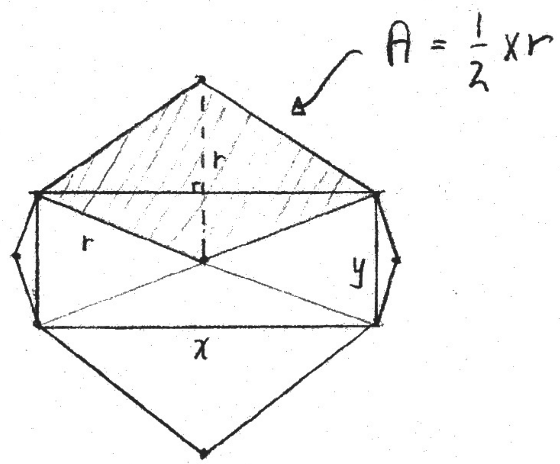
Appendix

Eight years later, I found a simpler way to calculate the area of the octagon, using $A = \frac{1}{2} d_1 d_2$ for a quadrilateral whose diagonals d_1 and d_2 are perpendicular. For example:



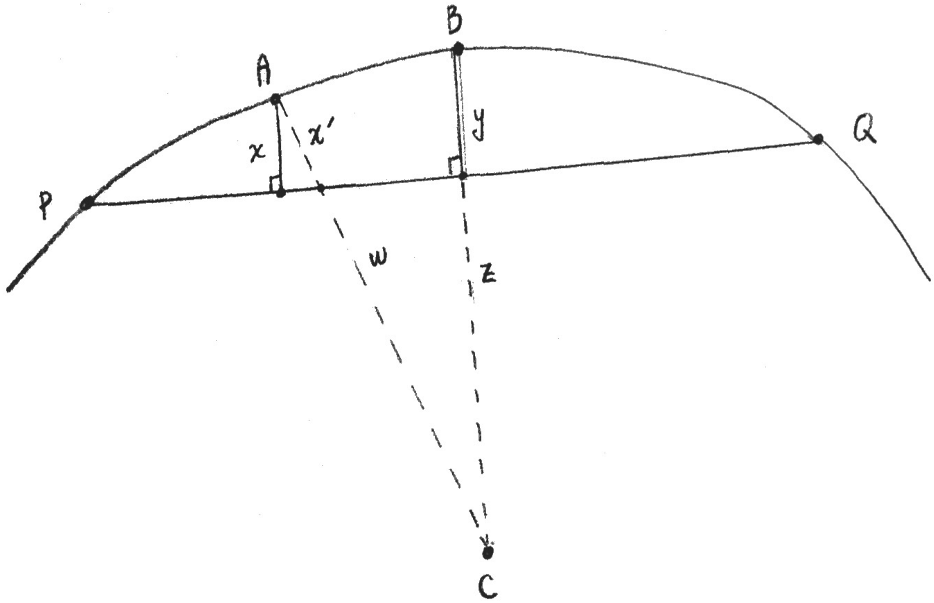
So the area of the octagon is:

$$2 \cdot \left(\tfrac{1}{2} x r\right) + 2 \cdot \left(\tfrac{1}{2} y r\right) = (x+y) \cdot r ,$$

as desired.

Long Pond, PA
4 October 2014

Another Appendix



$x < y$

$\Leftarrow$ { $x < x'$ because x is the shortest distance from A to $\overline{PQ}$ }

$x' < y$

$\equiv$ { algebra ; $x' + w = y + z$ }

$w > z$

$\equiv$ { z is the shortest distance from C to $\overline{PQ}$ }

true