Why we use strong steps

When we have $[\ P \ \Rightarrow\ Q\ ]$, we say that P is stronger than Q (or that Q is weaker than P). When we have a choice between proving a stronger theorem and proving a weaker theorem, clearly we prefer to prove the stronger one; the weaker one then follows as a logical consequence. It is important for a newcomer to the calculational style to understand that this principle guides us even when our theorems are the smallest they can be: namely, just one step in a calculation.

Suppose we are constructing a weakening chain with goal R , and we are at intermediate stage P . Further suppose we are faced with two choices for continuing our chain: the choices are $Qstrong$ and $Qweak$, with $[\ Qstrong \ \Rightarrow\ Qweak\ ]$. Which do we choose? Well, this is the choice between proving $[\ P \ \Rightarrow\ Qstrong\ ]$ and proving $[\ P \ \Rightarrow\ Qweak\ ]$. Thanks to the monotonicity of $\Rightarrow$ in its right argument, the former is stronger, so we continue the chain with $Qstrong$. Note that our remaining proof obligation is then $[\ Qstrong \ \Rightarrow\ R\ ]$, which is weaker than $[\ Qweak \ \Rightarrow\ R\ ]$, thanks to the antimonotonicity of $\Rightarrow$ in its left argument.

Similarly, if we are in a strengthening chain, then the choice is between $[\ P \Leftarrow Qstrong\ ]$ and $[\ P \Leftarrow Qweak\ ]$. Thanks to the antimonotonicity of $\Leftarrow$ in its right argument, the latter is stronger, so we continue the chain with $Qweak$. Note that our remaining proof obligation is then $[\ Qweak \ \Leftarrow\ R\ ]$, which is weaker than $[\ Qstrong \ \Leftarrow\ R\ ]$, thanks to the monotonicity of $\Leftarrow$ in its left argument.

I should have mentioned that all of the above also depends on the monotonicity of the everywhere operator, which just means that we can ignore it as far as monotonicity is concerned. (I ignored it so well that I forgot to mention it!)

So to summarize, we prefer our calculational steps to be as strong as possible, which means that in weakening chains, each new line should be as strong as possible, and that in strengthening chains, each new line should be as weak as possible.

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