On long sequences of composite numbers (revised)

All variables are of type natural. We are asked to show that there exist arbitrarily long sequences of consecutive composite numbers. Given the desired length of such a sequence, the sequence is completely determined by its first element. So if the desired length of our sequence is N , then we are looking for an x such that:

(0) $\langle \forall i :\ 0 \le i < N \ :\ \text{composite}.(x+i) \rangle$.

This is the simplest way I can think of to formalize our demonstrandum.

Let us calculate a bit with the term of (0) :

$\text{composite}.(x+i)$

$\equiv$ { definition of composite ; here $\sqsubseteq$ is “divides” }

$\langle \exists d :\ 2 \le d < x+i \ :\ d \sqsubseteq x+i \rangle$

$\Leftarrow$ { • assuming $2 \le i < x+i$, instantiation $d := i$ }

$i \sqsubseteq x+i$

$\equiv$ { property of $\sqsubseteq$ }

$i \sqsubseteq x$.

Well, that is nice, but we needed:

$2 \le i < x+i$

$\equiv$ { arithmetic }

$2 \le i \ \wedge\ 0 < x$

$\equiv$ { • assuming $0 < x$ }

$2 \le i$.

So we have a new constraint on x , $0 < x$. A little more distressing is the condition $2 \le i$, while we only have $0 \le i < N$. We resolve this by simply shifting the range in (0) over by 2 units, yielding $2 \le i < N+2$. Thus the first term in our desired sequence is now $x+2$, rather than x .

And now we can calculate, leaving the range of i implicit:

$\langle \forall i :: \text{composite}.(x+i) \rangle$

$\Leftarrow$ { above calculation }

$\langle \forall i :: i \sqsubseteq x \rangle$

$\equiv$ { property of **lcm** }

$\langle \mathbf{lcm}\ i :: i \rangle \sqsubseteq x$.

Thanks to the reflexivity of $\sqsubseteq$, we may take $\langle \mathbf{lcm}\ i :: i \rangle$ for x , noting that this choice satisfies the obligation $0 < x$. If this choice does not satisfy us, we may take our calculation a step further:

$\langle \mathbf{lcm}\ i :: i \rangle \ \sqsubseteq\ x$

$\Leftarrow$ { shrinking, since $\langle \mathbf{lcm}\ i :: i \rangle \ \sqsubseteq\ \langle \Pi i :: i \rangle$ }

$\langle \Pi i :: i \rangle \ \sqsubseteq\ x$

$\equiv$ { recalling $2 \le i < N+2$; introducing f , the factorial function }

$f.(N+1) \ \sqsubseteq\ x$.

So we may take $f.(N+1)$ for x , again noting that this satisfies $0 < x$. Certainly the factorial function may be more easily calculated than the least common multiple; but the latter will usually be much smaller than the former.

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We end our discussion with a note about one of our design decisions. In the first calculation, we instantiated an existential quantifier with i , when conceivably we could have chosen x as well, yielding:

(1) $\quad \text{composite}.(x+i) \quad \equiv \quad x \sqsubseteq i$.

As the reader may check, this choice would require us to adopt constraint $2 \le x$, and to shift the range of i to $1 \le i < N+1$. So far so good. But now the final calculation would run as follows:

$\langle \forall i :: \text{composite}.(x+i) \rangle$

$\Leftarrow$ { (1) }

$\langle \forall i :: x \sqsubseteq i \rangle$

$\equiv$ { property of **gcd** }

$x \ \sqsubseteq\ \langle \mathbf{gcd}\ i :: i \rangle$.

Wow! It appears we have found a new possibility for x , namely $\langle \mathbf{gcd}\ i :: i \rangle$. Sadly, we have not. The **gcd** of a range of at least two consecutive numbers is equal to 1 , and 1 will not do as a choice for x , since this choice does not satisfy $2 \le x$.

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I am quite pleased with this revision of JAW1 , which was far messier than it needed to be. I'm glad I accomplished something in the past two years!

(But unfortunately I have not yet learned to avoid having my address stick out on the last page.)

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