Achilles and the tortoise once again

We return to the problem discussed in JAW33 : Achilles and the tortoise each have $n \geq 1$ slips of paper. Both write the numbers 1 through $2*n$ in an arbitrary order on their slips: on each side of the slip appears exactly one number. Prove that the slips of Achilles and the tortoise can be put down such that all numbers from 1 through $2*n$ are shown on the up side (ie no turning over is necessary to see each number) .

The slips constitute a bag of unordered pairs of the numbers 1 through $2*n$, and placing the slips corresponds to making the pairs of this bag ordered. For the sake of argument, let us stipulate that the first number of an ordered pair/slip corresponds to the number on the up side of that slip. Then a proper placement of the slips corresponds to an ordering where each number from 1 through $2*n$ appears as the first number of some ordered pair.

Well this is a problem about graphs. We are given the graph G with nodes 1 through $2*n$, and with edgeset equal to the slips. The question is then: can we orient the edges of G so that every node has outdegree at least 1 ? Because the number of numbers and the number of slips are equal —viz $2*n$ — , this question is equivalent to: can we orient the edges of G so that every node has indegree and outdegree equal? And by standard graph theory, we have:

the edges of G can be oriented so that for every node,

indegree and outdegree are equal

$\equiv$ { }

the edges of G can be partitioned into cycles

$\equiv$ { }

every node of G has even degree .

Since every number has been written exactly twice, every node has degree 2 , which is even, so the answer to our question is yes.

Observe too that we have solved the following more general problem: Given some slips of paper, such that on each side of a slip appears exactly one number. Prove that every number appears an even number of times, precisely when the slips can be placed so that every number appears up as many times as it appears down.

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