Absolute value distributes over multiplication

Today at the Eindhoven Tuesday Afternoon Club, while attempting to prove a theorem, Tom Verhoeff suggested we use that the absolute value distributes over multiplication, in symbols:

(0) $|x * y| \;=\; |x| * |y|$.

We looked in vain for a nice proof of this, but as we were only minutes from our coffee break, we did not devote our full attention to the task.

Later in the day, Tom was looking through a basic mathematics book, where he found absolute value defined by:

(1) $|x| \;=\; \sqrt{x^2}$,

where for $x \geq 0$, $\sqrt{x}$ is defined to be the (unique) solution of:

(2) $y : \quad y \geq 0 \quad \wedge \quad y^2 = x$.

Here is one proof of (0) , using (1) :

$|x * y|$

= { (1) }

$\sqrt{(x * y)^2}$

= { algebra: associativity and symmetry of $*$ }

$\sqrt{x^2 * y^2}$

= { $x^2 \geq 0$ and $y^2 \geq 0$; $\sqrt{\;}$ property, see below }

$\sqrt{x^2} * \sqrt{y^2}$

= { (1) }

$|x| * |y|$.

In the remainder of this note, I prove the square root property used in the above proof, and show that (1) agrees with the more traditional calculational definition of absolute value, namely:

(3) $|x| \;=\; x \uparrow (-x)$.

First we show

(4) $\sqrt{x * y} \;=\; \sqrt{x} * \sqrt{y}$,

wherever the terms of this formula are well-defined —that is, precisely when $x \geq 0$ and $y \geq 0$ — . Given the definition of $\sqrt{\ }$, we prove (3) by showing that $\sqrt{x} * \sqrt{y}$ is a solution to $\sqrt{x * y}$'s defining equation, that is:

$\sqrt{x} * \sqrt{y} \ \geq \ 0$
$\Leftarrow$ { pos calculus }
$\sqrt{x} \geq 0 \quad \wedge \quad \sqrt{y} \geq 0$
$\equiv$ { definition of $\sqrt{\ }$ }
true ,

and

$(\sqrt{x} * \sqrt{y})^2 \ = \ x * y$
$\equiv$ { algebra }
$\sqrt{x}^2 * \sqrt{y}^2 \ = \ x * y$
$\equiv$ { definition of $\sqrt{\ }$ }
$x * y \ = \ x * y$
$\equiv$ { equality }
true .

And finally we show:

(5) $\quad x \uparrow (-x) \ = \ \sqrt{x^2}$.

We prove (5) by showing that $x \uparrow (-x)$ is a solution to $\sqrt{x^2}$'s defining equation. This means showing

(6) $\quad x \uparrow (-x) \ \geq \ 0$

and

(7) $\quad (x \uparrow (-x))^2 \ = \ x^2$.

As for (6) , we have:

$x \uparrow (-x) \ \geq \ 0$
$\Leftarrow$ { lattice theory }
$x \geq 0 \quad \vee \quad -x \geq 0$
$\equiv$ { $\geq$ is linear }
true .

As for (7) , we have:

$$(x \uparrow (-x))^2 \;=\; x^2$$
$\equiv$ { algebra, see below }
$$x \uparrow (-x) \;=\; x \quad \vee \quad x \uparrow (-x) \;=\; -x$$
$\equiv$ { $\geq$ is linear }
true .

The property used in this last calculation is:

(8) $a^2 = b^2 \quad\equiv\quad a = b \;\vee\; a = -b$,

which follows from the 0 property of fields (in fact of integral domains):

$$a^2 = b^2$$
$\equiv$ { algebra }
$$a^2 - b^2 \;=\; 0$$
$\equiv$ { algebra }
$$(a - b) * (a + b) \;=\; 0$$
$\equiv$ { algebra: 0 property }
$$a - b \;=\; 0 \quad \vee \quad a + b \;=\; 0$$
$\equiv$ { algebra }
$$a = b \quad \vee \quad a = -b$$.

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