An incomplete introduction to Wim Feijen's proof format

In high school algebra, one often writes a derivation like the following:

$$2*x + 1 = 3$$
$$2*x = 2$$
$$x = 1 \qquad ,$$

and it would never occur to anyone to write this derivation thus:

$$2*x + 1 = 3$$
$$2*x = 2$$
$$2*x = 2$$
$$x = 1 \qquad ,$$

because in this way we have needlessly repeated the intermediary line $2*x = 2$.

Yet traditional mathematicians do this all the time in informal arguments:

> " Given $2*x+1 = 3$, we subtract 1 from both sides to yield $2*x = 2$. Now taking $2*x = 2$, we divide both sides by 2 to yield $x = 1$, whence the problem is solved. "

(Of course, traditional mathematicians would not be discussing such a trivial example in such detail, but the point stands.)

Because in the calculational style we wish to carry out proofs by massaging formulae, and because the intermediary formulae can be quite long, we consider it crucial to follow the high school teachers and use a proof format that allows us to eliminate repetition of the intermediate expressions.

Dutch mathematician and computing scientist Wim Feijen has devised such a format, whose steps are as follows:

$$P$$
$$\% \qquad \{ \text{ hint why } P \,\%\, Q \ \}$$
$$Q \qquad ,$$

where P and Q are boolean expressions and $\%$ is one of the following connectives:

$\equiv$ "equivales"

$\Rightarrow$ "implies", "requires"

$\Leftarrow$ "if", "follows from" .

We can string these steps together, like so:

$$
\begin{array}{ll}
 & P \\
\% & \{ \text{ hint why } P \% Q \ \} \\
 & Q \\
\# & \{ \text{ hint why } Q \# R \ \} \\
 & R \qquad ,
\end{array}
$$

and so forth. Stringing them together is meaningful because:

from $P \equiv Q$ and $Q \equiv R$ we conclude $P \equiv R$,

from $P \equiv Q$ and $Q \Rightarrow R$ we conclude $P \Rightarrow R$,

from $P \equiv Q$ and $Q \Leftarrow R$ we conclude $P \Leftarrow R$,

from $P \Rightarrow Q$ and $Q \Rightarrow R$ we conclude $P \Rightarrow R$,

from $P \Leftarrow Q$ and $Q \Leftarrow R$ we conclude $P \Leftarrow R$, etc.

In other words, with this proof format we can treat the boolean operators just like we treat the relations $=$, $\leq$, and $\geq$ on reals.

Also please notice how the proof format has three levels of indentation: the connectives, the hints, and the expressions are all lined up.

Books could be written about the role of the hints. We say nothing here about them, except that they should be afforded at least a full line each, so they can be as clear and as informative as possible.

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Returning to the example at the start of this note, we would write this derivation in Wim Feijen's proof format as follows:

$$
\begin{array}{ll}
 & 2 * x + 1 = 3 \\
\equiv & \{ \text{ algebra: subtract } 1 \text{ from both sides } \} \\
 & 2 * x = 2 \\
\equiv & \{ \text{ algebra: divide both sides by } 2 \ \} \\
 & x = 1 \qquad .
\end{array}
$$

This might seem like overkill. I disagree! Few high school teachers and probably no high school students realize that the logical connective between each line is the equivalence, or even that there's a logical connective there at all. And what a disservice it does to hide that connective from students: it prevents them from knowing what they're really doing in these derivations, namely, simplifying a boolean expression without changing its value.

Indeed, most students I've worked with have no idea what they're doing when they "solve for x", and it only becomes harder for them to learn it later. Let us be explicit, and introduce students to the boolean domain with Wim Feijen's elegant proof format!

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