Local separation of concerns

An important technique in the calculational style is "separation of concerns" ; that is, the isolation of different facets of a problem, so they can be dealt with separately. In this note, I demonstrate how this technique can be applied even on the level of symbols.

We are requested to show:

$$(0) \qquad m^2 \geq n^2 \quad \Leftarrow \quad m \geq n \qquad ,$$

for all positive, real m and n . One proof of (0) is as follows:

$m^2 \geq n^2$

$\equiv$ { definition of ^{2} }

$m*m \geq n*n$

$\Leftarrow$ { transitivity of $\geq$ }

$m*m \geq m*n \quad \wedge \quad m*n \geq n*n$

$\equiv$ { cancelling, using $m, n > 0$ }

$m \geq n \quad \wedge \quad m \geq n$

$\equiv$ { idempotence of $\wedge$ }

$m \geq n$.

Let us walk through this proof. The first step embodies the design decision behind the entire calculation: Not knowing the desired property (monotonicity) of exponentiation, we choose to unfold it in terms of multiplication, which does satisfy some monotonicity properties.

The second step, which exploits the transitivity of $\geq$, is a bit of a mystery. As the remaining steps are completely standard simplifications, we focus on that second step.

The formula $m*m \geq n*n$ exhibits a symbolic lack of separation of concerns. Indeed, there is a difference of two symbols between the left and righthand side: On one side of the equation, we have two m's , on the other, two n's . So that we can deal with one difference at a time, we exploit the transitivity of $\geq$ and bridge the gap with the intermediate expression $m*n$. (For the experienced calculator, this sort of step is standard; see, for example, the proof on page 19 of EWD1300 , by Edsger Dijkstra.)

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This observation also gives us some constructive ways to complify. In JAW2 , for example, I remarked that it would be silly to try to prove $x^2 - y^2 = (x-y)*(x+y)$

by transforming $x^2 - y^2$ into $(x-y)*(x+y)$, because in so doing one has to pull the term $x*y$ out of the proverbial magic hat. I still largely agree with this remark, because the term $(x-y)*(x+y)$ immediately invites massaging and simplification, but I will now add that if we didn't know a factorization of $x^2 - y^2$, we could derive it, using the principle of separation of concerns:

$x^2 - y^2$

= { Because there is a difference of two symbols between x^2 and y^2 , it would be nice to introduce the intermediate term $x*y$. We can do this by adding and subtracting it, thereby leaving the value of the original expression unchanged. }

$x^2 + x*y - x*y - y^2$

= { The terms $x*y$ allow us to factor x^2 and y^2 . }

$x*(x+y) - y*(x+y)$

= { Factoring once again. }

$(x-y)*(x+y)$.

What is nice about this derivation is that although every step except the last is "experimental" , each step is well-chosen and sweetly reasonable, because it is guided by the shape of the formulae.

As an exercise, the reader might enjoy attempting a "naive" factorization of $x^3 - y^3$ and $x^3 + y^3$. (The idea, as above, is to use algebra to introduce "stepping stones" between x^3 and y^3 .)

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