Case analysis from a calculational perspective

Introduction

The fuzzy, human notion of ‘proof’ has a crisp, completely precise correlate, namely the notion of ‘contextual calculation’ . Here I wish to say a few words about the proof strategy known as ‘case analysis’ in terms of contexts and calculations. The picture I wish to present is that case analysis is a means of strengthening a context, at no cost.

To illustrate the principle, I will consider a case analysis into two cases.

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“Local calculations”

Suppose our goal is to calculate X , in a context C . We put C into what is called the “global context” , and draw on it as necessary. All of our calculations have an implied antecedent of C .

Now suppose we see a chance to calculate X using an additional property P . (The stronger the context, the weaker the proof obligation, thanks to $\Rightarrow$ ’s antimonotonicity in its left argument.) So we set up a “local context” containing P , and then make our calculation drawing on P (as well as the global context C) . We call this sort of calculation a “local calculation” , and the result of it may be described as:

(0) $P \Rightarrow X$.

Suppose we also see a chance to calculate X using an additional property Q . Following the same technique above, we make another local calculation, equivalent to:

(1) $Q \Rightarrow X$.

Altogether, we have calculated the conjunction of (0) and (1) , by predicate calculus equivalent to:

(2) $P \vee Q \Rightarrow X$.

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Relating to the original calculandum

However, $P \vee Q \Rightarrow X$ is weaker than X, the original calculandum. So the only way to salvage what we have done is to relate P and Q to C somehow.

Now is the time to remember that (2) was calculated in the context of C. Thus we bring the global context back into the picture in order to calculate with it:

$C \Rightarrow (P \vee Q \Rightarrow X)$

$\equiv$ { predicate calculus }

$C \wedge (P \vee Q) \Rightarrow X$

$\equiv$ { predicate calculus, using $C \Rightarrow P \vee Q$ }

$C \Rightarrow X$

$\equiv$ { }

"the original calculandum" .

So provided $C \Rightarrow P \vee Q$, the conjunction of the local calculations is equivalent to the original one.

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Conclusion

The technique of case analysis may thus be summarized as follows: The main calculation can be accomplished via a series of local calculations, provided:

(3) the global context

$\Rightarrow$

the disjunction of the local contexts .

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Examples

Often (3) is met without use of the global context, by making the disjunction of the local contexts **true** , say by taking R and $\neg R$ for the local contexts.

But this is not always the case. For instance, if

$$-2 \leq x < 7$$

is a conjunct of the global context, then we may take for instance:

$$-2 \leq x < 5 \quad \text{and} \quad 5 \leq x < 7$$

for the local contexts. (This is known as range splitting.)

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Final note

None of this can really be said to be new. But I think it is helpful to see how notions like case analysis become so crisp when phrased in terms of crisp notions like contexts, calculations, and predicate calculus. (Once again an example of how interfaces make all the difference.)

I challenge my readers to come up with an explanation of case analysis clearer than (3) , without using this conceptual interface.

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Jeremy Weissmann
11260 Overland Ave. #21A
Culver City, CA 90230
USA
jeremy@mathmeth.com