Not about primality

We are asked to show that every natural number ≥ 7 is a sum of distinct primes.

Our first question in any existence proof should be: "Should we construct a witness?" . In this case, the question becomes: "Should we give an algorithm for finding the distinct primes which sum to a given number?" .

I think the most reasonable answer here is no. We give this answer because in the problem statement, we are asked to find primes where nothing about primality seems relevant: our simplest tools for dealing with primality involve multiplication, not addition.

Thus we conjecture not only that a nonconstructive solution will be easier, but also that little if nothing about primality may be relevant. Aiming for a disentangled argument, where we derive the relevant properties of primality, we introduce the notion of "florg" natural numbers, and write:

$$f.n \quad \equiv \quad \text{“ } n \text{ is a florg ”} \qquad .$$

Our problem statement thus becomes: Show that

(0) every natural number ≥ 7 is a sum of distinct florgs .

* * *

So far, so good. Now, how can we establish (0) ? Since (0) is about natural numbers, we could use induction, or perhaps prime factorization. However, since we already decided that (0) is not about primality, it seems more reasonable to use induction. But let's make sure we're not overlooking anything: Does anything else suggest induction?

In fact, induction seems tailor-made to the structure of our argument, thanks to:

$$\begin{array}{lll} & & k \text{ is a sum of distinct florgs } < p \\ (1) & \Rightarrow & \{ \text{ adding florg } p \ \} \\ & & k+p \text{ is a sum of distinct florgs } \leq p \qquad . \end{array}$$

Doing it for larger numbers, provided you can do it for smaller numbers, is what induction is all about!

Now think of the structure of an induction argument establishing (0) : For the induction step, we are given that all numbers up to a certain point are sums of distinct florgs. By adding a fresh florg to all these sums, we create larger sums of distinct florgs. (Here we are borrowing the idea that there are always fresh florgs in the sea of natural numbers!) We just have to take care that we can always find the right florgs to make this idea work, and so we make this idea more precise.

Here is one possible formalization of the above idea. Given a florg p , we wish to establish the following step:

$$
\begin{array}{lll}
 & & \langle \forall k \ : \ 7 \leq k \leq n \ : \quad k \ \text{ is a sum of distinct florgs } \ < p \ \rangle \\
(2) & \Rightarrow & \quad \{ \ \text{adding } \ p \ \text{ to each number in the range } \} \\
 & & \langle \forall k \ : \ 7 \leq k \leq n{+}p \ : \quad k \ \text{ is a sum of distinct florgs } \ \leq p \ \rangle \qquad .
\end{array}
$$

With (2) , we can establish the following induction argument:

$$
\begin{array}{ll}
 & \langle \forall k \ : \ 7 \leq k \leq n \ : \quad k \ \text{ is a sum of distinct florgs } \ \leq F.n \ \rangle \\
\Rightarrow & \quad \{ \ \bullet \ \text{ using } \ F.n \ < \ p \ \} \\
 & \langle \forall k \ : \ 7 \leq k \leq n \ : \quad k \ \text{ is a sum of distinct florgs } \ < p \ \rangle \\
\Rightarrow & \quad \{ \ (2) \ \} \\
 & \langle \forall k \ : \ 7 \leq k \leq n{+}p \ : \quad k \ \text{ is a sum of distinct florgs } \ \leq p \ \rangle \\
\Rightarrow & \quad \{ \ \bullet \ \text{ using } \ p \ \leq \ F.(n{+}p) \ \} \\
 & \langle \forall k \ : \ 7 \leq k \leq n{+}p \ : \quad k \ \text{ is a sum of distinct florgs } \ \leq F.(n{+}p) \ \rangle \qquad .
\end{array}
$$

It only remains to design F and p satisfying conditions (2) and:

$$
\begin{array}{ll}
(3) & F.n \ < \ p \\
(4) & p \ \leq \ F.(n{+}p) \qquad .
\end{array}
$$

(For the above to be a valid induction argument, we also need $n{+}p \ > \ n$, which equivales $p > 0$, hence we assume about f :

$$
f.p \ \Rightarrow \ p > 0 \qquad \text{for all } \ p \qquad .)
$$

*   *
*

So far, so very good. The next step is to prove (2) in detail, so as to derive more conditions on p and n . We start with the more complex consequent, and construct a strengthening chain to the antecedent. In the following proof, we suppress a bit of the internal structure of our predicate, and write "...." for "is a sum of distinct florgs" :

$$
\begin{array}{ll}
 & \langle \forall k \ : \ 7 \leq k \leq n{+}p \ : \ k \ \ldots. \ \leq p \ \rangle \\
\equiv & \quad \{ \ \text{range split, to get closer to the antecedent } \} \\
 & \langle \forall k \ : \ 7 \leq k \leq n \ : \ k \ \ldots. \ \leq p \ \rangle \quad \wedge \quad \langle \forall k \ : \ n{+}1 \leq k \leq n{+}p \ : \ k \ \ldots. \ \leq p \ \rangle \qquad .
\end{array}
$$

We strengthen each conjunct into the antecedent. First the left:

$\langle \forall k : 7 \le k \le n : k \;....\; \le p \rangle$

$\Leftarrow$ $\{$ "$\le p$" $\Leftarrow$ "$<p$" $\}$

$\langle \forall k : 7 \le k \le n : k \;....\; < p \rangle$,

then the right:

$\langle \forall k : n+1 \le k \le n+p : k \;....\; \le p \rangle$

$\Leftarrow$ $\{$ our original observation (1) $\}$

$\langle \forall k : n+1 \le k \le n+p : k-p \;....\; < p \rangle$

$\equiv$ $\{$ transforming the dummy with $k := k+p$ $\}$

$\langle \forall k : n+1 \le k+p \le n+p : k \;....\; < p \rangle$

$\equiv$ $\{$ subtracting p from the range $\}$

$\langle \forall k : n+1-p \le k \le n : k \;....\; < p \rangle$

$\Leftarrow$ $\{$ • widening the range, using $7 \le n+1-p$, equivalently $p \le n-6$ $\}$

$\langle \forall k : 7 \le k \le n : k \;....\; < p \rangle$.

Thus we have proved (2) , at the cost of a final condition on p , namely:

(5) $p \le n-6$.

* * *

Now we turn to conditions (3)–(5) . Conditions (4) and (5) are suspiciously similar, which allows us to derive a candidate for F :

$p \le n-6$

$\equiv$ $\{$ adding p , to create $n+p$ in the righthand side $\}$

$2*p \le n+p-6$

$\equiv$ $\{$ dividing by 2 , restoring p in the lefthand side $\}$

$p \le (n+p-6)/2$.

Guided by this computation, we define:

$$F.n \quad = \quad (n-6)/2 \qquad ,$$

whence our three properties reduce to:

(6) $\quad (n-6)/2 \ < \ p \ \leq \ n-6 \qquad .$

For those in the know, we have just reinvented the so-called Bertrand's Postulate , which in our florg terminology states that for any real $r \geq 2$, there exists a florg p satisfying

$$r \ < \ p \ < \ 2*r \qquad .$$

Taking Bertrand's Postulate as an assumption about florgs, condition (6) reduces to:

$$\begin{array}{ll} & (n-6)/2 \ \geq \ 2 \\ \equiv & \quad \{ \text{ algebra } \} \\ & n \ \geq \ 10 \qquad , \end{array}$$

and the proof of the induction step is complete.

* * *

The final part of our argument is the "base case" : In other words, we need some natural $n \geq 10$, which satisfies:

$$\langle \forall k \ : \ 7 \leq k \leq n \ : \quad k \ \text{ is a sum of distinct florgs } \ \leq \ F.n \ = \ (n-6)/2 \ \rangle \qquad .$$

The reader can easily check that the smallest such n is 32 , under the assumption that 2 , 3 , 5 , 7 , 11 , and 13 (that is, $F.32 = (32-6)/2$) are all florgs.

* * *

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