A bit on predicate splitting, and an exploration

Predicate splitting

In JAW65 I briefly discussed the notion of predicate splitting. Let me recall that discussion here.

When we say we are "splitting" a predicate X , this means we are finding predicates P and Q such that:

(0) $\quad P \wedge Q \quad \Rightarrow \quad X \qquad .$

Note that (0) is equivalent by shunting to $Q \Rightarrow \neg P \vee X$. Thus for any P , the weakest choice for Q is $\neg P \vee X$. Hence the "canonical split" of a predicate X is into P and $\neg P \vee X$, for fresh P .

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An error

For certain predicates, there may be splits which suggest themselves. For instance, if our predicate is $x = y$ in the reals, this suggests a natural split into $x \leq y$ and $y \leq x$. Now, adopting the nomenclature above, $X \equiv x = y$ and $P \equiv x \leq y$, so the weakest choice for Q is:

$\quad \neg P \ \vee \ X$

$\equiv \quad$ { see above }

$\quad \neg(x \leq y) \ \vee \ x = y$

$\equiv \quad$ { arithmetic }

$\quad y < x \ \vee \ x = y$

$\equiv \quad$ { arithmetic }

$\quad y \leq x \qquad .$

So in this case, the obvious split was in fact the canonical split.

When we make observations (like the above), it is natural to try to generalize by analogy. However, it is not uncommon to make an error while doing so! And this is exactly what happened to me, after making the above generalization.

I now considered an arbitrary antisymmetric relation $\mathcal{R}$, and reasoned as follows: Because $\mathcal{R}$ is antisymmetric, we can split $x = y$ into $x\mathcal{R}y$ and $y\mathcal{R}x$. So then, as before, we must have:

(1) $\neg(x\mathcal{R}y) \quad \vee \quad x = y \qquad \equiv \qquad y\mathcal{R}x \qquad .$

Hopefully the analogy to the previous example is clear, but can you spot the error?

Our discussion of predicate splitting does *not* tell us that for every split of X into P and Q, we have $Q \equiv \neg P \vee X$, only that the *weakest* choice for Q is $\neg P \vee X$. So all we can conclude in this general example is:

(2) $\neg(x\mathcal{R}y) \quad \vee \quad x = y \qquad \Leftarrow \qquad y\mathcal{R}x \qquad ,$

which is just a restatement of $\mathcal{R}$'s antisymmetry.

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An investigation

Apurva Mehta, who witnessed my intellectual thrashing, decided to seek out a sufficient condition for (1) . Of course, he did this by calculating, and I will try to repeat his lovely calculation by heart. First of all, half of (1) is (2) , which is simply a restatement of $\mathcal{R}$'s antisymmetry. Hence it is sweetly reasonable to establish (1) by a ping-pong argument, whence our remaining calculandum is:

$\neg(x\mathcal{R}y) \quad \vee \quad x = y \qquad \Rightarrow \qquad y\mathcal{R}x$

$\equiv$ { predicate calculus }

$(\neg(x\mathcal{R}y) \quad \Rightarrow \quad y\mathcal{R}x) \qquad \wedge \qquad (x = y \quad \Rightarrow \quad y\mathcal{R}x)$

$\equiv$ { predicate calculus }

$(x\mathcal{R}y \quad \vee \quad y\mathcal{R}x) \qquad \wedge \qquad (x = y \quad \Rightarrow \quad y\mathcal{R}x)$

$\equiv$ { " for all x, y " }

$\mathcal{R}$ is linear and reflexive .

So we have proved that (1) is equivalent to $\mathcal{R}$ being reflexive, antisymmetric, and linear.

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