Simplifying inequalities with Wim

While going through old solution sets I had written for a high school calculus class, I came across the following exercise: Find all x satisfying:

(0) $|x-1| \ + \ |x-2| \ > \ 1$.

In my original solution, I split the problem into three cases: $2 < x$, $x < 1$, and $1 \leq x < 2$. Now I see why that case analysis was self-inflicted, thanks to the traditional definition of absolute value:

$$|x| \ = \ \begin{cases} x\,, & \text{if } x \geq 0 \quad ; \\ -x\,, & \text{if } x < 0 \quad . \end{cases}$$

I have since learned from Wim the joys of calculating with absolute value, once we use our superior definition of absolute value:

(1) $|x| \ = \ x \uparrow (-x)$.

This definition is lovely, because $\uparrow$ (the binary maximum operator) has many nice properties, two of which are:

(2) $(x \uparrow y) + z \quad = \quad (x+z) \uparrow (y+z)$

(3) $x \uparrow y \ > \ z \quad \equiv \quad x > z \ \vee \ y > z$.

With these properties in hand, we can straightforwardly calculate with (0) :

$|x-1| \ + \ |x-2| \ > \ 1$

$\equiv$ { (1) , unfolding absolute value }

$(x-1) \uparrow (-x+1) \ + \ (x-2) \uparrow (-x+2) \quad > \quad 1$

$\equiv$ { (2) , distributing $+$ over $\uparrow$ completely }

$(x-1+x-2) \uparrow (x-1-x+2) \uparrow (-x+1+x-2) \uparrow (-x+1-x+2) \quad > \quad 1$

$\equiv$ { algebra }

$(2\cdot x-3) \uparrow 1 \uparrow (-1) \uparrow (-2\cdot x+3) \quad > \quad 1$

$\equiv$ { (3) }

$2\cdot x-3 > 1 \quad \vee \quad 1 > 1 \quad \vee \quad -1 > 1 \quad \vee \quad -2\cdot x+3 > 1$

$\equiv$ { algebra }

$x > 2 \ \vee$ **false** $\vee$ **false** $\vee \ x < 1$

$\equiv$ { predicate calculus }

$x > 2 \ \vee \ x < 1$.

That is lovely, I think! I carried out this derivation in such detail because I wanted to show that our hand was forced at each step.

* * *

Warning. Please note that our original formula (0) was of the form:

$$|x| + |y| \; > \; c \qquad ,$$

which could be expanded into:

$$\begin{array}{ll} & +x+y>c \\ \vee & +x-y>c \\ \vee & -x+y>c \\ \vee & -x-y>c \qquad . \end{array}$$

But don't be fooled into thinking we have something like

(4) $\quad P.|x| \;\equiv\; P.x \;\vee\; P.(-x) \qquad ,$

for all predicates P , as this would yield erroneous results like:

$$\begin{array}{ll} & |x| < 0 \\ \equiv & \{ \;\; (4) \;\; \} \\ & x < 0 \;\;\vee\;\; -x < 0 \\ \equiv & \{ \text{ algebra } \} \\ & x \neq 0 \qquad , \end{array}$$

when we should actually have:

$$\begin{array}{ll} & |x| < 0 \\ \equiv & \{ \;\; (1) \;\; \} \\ & x \uparrow (-x) \; < \; 0 \\ \equiv & \{ \text{ another property: } \; a \uparrow b \; < \; c \quad \equiv \quad a < c \;\wedge\; b < c \;\; \} \\ & x < 0 \;\;\wedge\;\; -x < 0 \\ \equiv & \{ \text{ algebra } \} \\ & \textbf{false} \qquad . \end{array}$$

(**End** of Warning.)

* *
*

Deep thanks go to Allison Largent for catching a mistake in an earlier draft:

$$-2\cdot x+3 \ > \ 1 \quad \equiv \quad x<2 \qquad ,$$

which had allowed me to erroneously conclude $(0) \ \equiv \ x \neq 2$.

Much later, on 5 May 2006, Dan Grundy caught a terrible mistake in formula (3) .

The Cave, Santa Cruz, 1 November 2004

Revised: NYC, 28 December 2010

Jeremy Weissmann
jeremy@mathmeth.com