We shall overcome (some of our interfaces with boolean concepts)

Introduction

This note is about interfaces, in particular our interfaces with boolean concepts. In the first half of this note, I introduce all the relevant terminology. In the second half, I illustrate my points with some mathematical examples. The halves are separated by a pattern of three stars for the reader's convenience.

On interfaces

If two Things do not interact directly, they are said to interact through an **interface** . In this note we focus on interfaces where one of the Things is a person, and where the other Thing is just called a Thing and we don't care what it is. Thus we may speak of people **using** interfaces (with Things).

One benefit to using interfaces is that the interface may be more pleasant to deal with than the Thing itself. For example, many people prefer to make purchases using a bank card (and the underlying bank system) instead of with cash directly. And many people prefer to program in a high-level language like C++ rather than in machine code, or through electrical engineering. Scientific models are interfaces, too: a theory of gravity, for instance, helps us to have an understanding of motion and friction. And so forth.

An interface provides a way of doing, or a way of understanding. The above interfaces are, for the most part, fairly pleasant.

There are unpleasant interfaces too —sometimes these are called "middle men" — , which make for unpleasant ways of doing and understanding. Rather than interacting with the Thing directly, you have to first navigate the awkward interface. For instance, some children learn to add and subtract integers via the number line, or through some other visualization, like dumptrucks digging and filling holes; this is their (only) way of understanding the addition and subtraction of integers. The consequence is that each time they wish to add or subtract, they first have to conjure up an irrelevant and possibly confusing mental picture, when it would be simpler to just do the mathematical computation!

As another example, imagine that one of your colleagues uses a secretary to screen their calls. Each time you wish to speak to that colleague you have to speak to their secretary first, even though you would prefer to call the colleague directly. So for you, this secretary is an unpleasant interface. But let's look at it from another perspective: If your colleague has a lot of (possibly irate) callers, and a good secretary, they might find their secretary to be a very pleasant interface indeed! Which just goes to show that the utility of an interface depends on the needs of its user.

As a final example to show how useful interfaces can be, observe that the very notion of an interface can be seen as an interface between ourselves and our perceptions of Things. When we look at the world, we see Things interacting, and we can understand these interactions in terms of interfaces.

More generally, all human concepts may be thought of as interfaces between ourselves and our perceptions. In the next section we look at the notion of a concept in more detail.

On concepts and thought

Most of my readers will be familiar with those peculiar mental objects called **concepts** (or ideas) . One way of viewing a concept is as a cluster of properties. Most of the concepts we encounter have an enormous number of properties. For instance, consider a red apple. A red apple is an apple, red on the outside, white on the inside, sort of round but actually peculiarly apple-shaped, and small enough to hold in the hand. It is sweet and crunchy, has seeds, grows on trees, is a part of nature, has a stem on one side, a leafy navel on the other, can be bought in a market for less than a dollar, and so on. In short: we know more about a red apple than we could ever hope or want to articulate.

And yet our subconscious mind deals with this mass of properties almost effortlessly. We never feel overwhelmed by all the things we know about an apple, but rather our subconscious mind picks out the relevant information almost instantaneously, as the need arises. Subconscious thought thrives on rich, property-laden concepts.

By comparison, conscious thought is a weakling, barely capable of juggling a small handful of concepts or properties without getting confused or making an error. So when we use conscious thought to reason about a complex concept, we form an interface with that concept by choosing just a small handful of its properties to reason about. This smaller collection of properties forms a new concept, an **abstraction** from the original concept. We reason in terms of the abstract concept, and then apply our conclusions back to the original concept.

For example, we never reason about a red apple by considering all the things we know about it at once: why would we care about its color and its price at the same time? Perhaps today we are interested in its color alone, so we abstract from the concept of a red apple to form the more abstract concept of 'red' . Anything we conclude about 'red' will be applicable to a red apple, because a red apple is red!

We may form many different interfaces with the same concept. For instance, we may think of humans in terms of biology, in terms of physics, in terms of romance, in terms of physical appearance, etc, each corresponding to one of the many facets of humans that we are interested in. Each interface has its own terminology, and serves a different purpose. For example, the physical interface may not be very useful for reasoning about romance, and vice versa!

Also, please observe that these ways of understanding concepts, subconscious thought, and conscious thought are all just interfaces. They are not "proven facts" , nor do they require scientific verification: they are simply ways of understanding.

Boolean concepts

Among the most important concepts are the **boolean** concepts, the concepts relating to **truth** . Truth is our way of understanding how things will be. So truth is an interface, an abstraction from our experiences, observations, generalizations, and conclusions. This view may seem strange to some, since truth is often thought to be solid, definite, and unchanging. But remember that interfaces with concepts have nothing to do with what might be called "reality" ; they are only ways of understanding. The better an understanding we have of truth, the more accurately we will predict how things will be, and the more able we will be to achieve our desires.

Let me give a small example. Above I mentioned that we have a physical interface for understanding humans. Well, part of that interface tells us that the notion "we fall pretty fast" is a useful way of thinking about the physics of humans. So our concept of truth includes this notion, which we call a **fact** . If this were the only fact we knew about falling, we might conclude that we should jump out of a tall building to save time.

However, our physical interface also tells us that "the higher you fall from, the more it hurts" is a useful way of thinking about the physics of humans. So under advisement of this interface, it is also an unfortunate **fact** that jumping out of a tall building will severely wound us, if not kill us. Putting our facts together, we have a useful way to look at the world. Indeed, most people who jump out of tall buildings are severely wounded or killed, and rather quickly.

However, there are rare cases of people who have fallen hundreds of feet and survived, even without breaking a single bone. I mention this not to suggest that my readers should jump out of tall buildings after all, but just to point out that truth can be inaccurate. Like any interface, it is only a way of understanding. It is no more than that, and it doesn't need to be.

As in the previous sections, this way of understanding truth is just an interface.

Causality

Before we get to the mathematical content of this note, I would like to touch on the issue of **causality** . Causality is one way of connecting perceptions which are in principle disparate. We see a person move their finger, we hear a loud noise, and a split second later another person collapses to the floor in pain, bleeding from a small hole. But in our minds, the first person is not simply moving their finger, but causing a gun to fire, and having appropriately aimed that gun, has caused the second person to be wounded. Clearly causality is an important part of our understanding of truth, because it allows us to explain why certain things happen, and to draw conclusions. Thus it is a fundamental aspect of conscious thought.

However, to our detriment, it is also a fundamental aspect of subconscious thought. (For no reason that I can think of.) And this can often cloud our conscious thought, because our subconscious mind is all too eager to seek out causality. For instance, if someone is presented with two slides, the first showing a tall rectangle, the second showing a wide rectangle (of the same dimensions), they may describe what they have seen by saying: "It fell." . Without realizing it, they have created an anthropomorphic interface with their perceptions.

Understanding the connection between causes and effects is almost always a subtle matter, too important to be trusted to the subconscious. But since subconscious thought is far more ubiquitous than conscious thought, our minds can easily become cluttered with false connections.

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Consequences: Whatever happened to boolean equality?

Many people are uncomfortable with the notion of **boolean equality** , or boolean sameness. This is surprising, since in other domains equality seems to be one of the most natural concepts: Two small children are playing, the first with an apple and two oranges, the second with an apple and two pears. They will easily recognize that they have the same number of apples, and also the same number of fruits.

But in the boolean domain, equality is not a very natural concept. Thanks to the ubiquity of causality in thought, it is much more natural to think in terms of **implication** . Thus the equality "A equals B" is turned into a mutual implication, or biconditional: "A implies B, and B implies A" . In mathematics, where equality is even more important, it is very rare to see " $=$ " in print for boolean equality. It is much more common to find " $\Leftrightarrow$ " , which shows the implication " $\Rightarrow$ " pointing in both directions.

Can you imagine doing this with numbers? Can you imagine one of the children thinking this way: "Do we have the same number of fruits? Well, for that to happen, I should have at least as many fruits as them, and they should have as many fruits as me. I have three fruits and so do they, so I have as many fruits as them. And they have three fruits and so do I, so they have as many fruits as me. So we have the same number of fruits." ?

It is true that in the domain of numbers, "A equals B" is equivalent to "A at least B, and B at least A" ; in other words, there are two ways of understanding —two interfaces for— numerical equality. When we approach a problem, we need to choose the most appropriate interface, the right tool for the job. In the above example, clearly the latter interface is a very poor choice!

In the boolean domain, there are also two ways of understanding equality, but the simpler one —just flat out equality— is little known, thanks to a quirk of our brains. This has led to some very ugly reasoning, and some very ugly mathematics. I implore my readers to try to avoid short-changing boolean equality in the future.

Consequences: Whatever happened to “follows from” ?

Another consequence of the ubiquity of causality is that we become used to understanding implications in only one direction, $\Rightarrow$. In symbols: given boolean A , people feel at ease finding B's such that $A \Rightarrow B$, but feel uneasy finding B's such that $A \Leftarrow B$. (As with the case of boolean equality, the symbol “ $\Leftarrow$ ” for “follows from” is rarely found in print.) If you give someone an implication of the shape $B \Leftarrow A$, they will very often try to understand it by putting it first in the more “natural” direction, $A \Rightarrow B$.

But this is a shame, firstly because $\Leftarrow$ is the direction of abstraction, generalization, and creativity, and secondly because often it is more convenient to reason “backwards” than “forwards” . For example, when there is a problem to be solved, sometimes the best way to approach it is to analyze the problem itself, ask what you would need to solve it, and then work backwards from there. This technique is especially useful when you are given so much information that you have no idea where to begin. I hope that my readers have had such an experience while trying to solve a math problem or a puzzle.

Consequences: What is an implication about?

This last consequence is related to the previous one. It is hard to articulate exactly what this consequence is, but I think it has something to do with the notion that an implication $A \Rightarrow B$ is “about” B .

For example, consider the property of a function f being “injective” . This property could be written:

(0) for all x, y : $f.x = f.y \quad \Rightarrow \quad x = y$.

But this mathematical formalization is rare. Might this be because the implication is “about” $x = y$, and wasn't this supposed to be a property of f ?

Much more common is the equivalent formalization:

(1) for all x, y : $x \neq y \quad \Rightarrow \quad f.x \neq f.y$.

Ah, much better: now we can conclude something about f . The English language transcription of (1) is also very common, namely:

> “Different arguments have different function values.” .

But the following transcription of (0) I have never seen in print:

> “Equal function values have equal arguments.” .

(Maybe this suggests that equality is less natural than I thought.)

As a final example, a colleague of mine was recently exploring a property of numbers I will call **blorg** . Through some investigation, he discovered:

(2) for all n : **blorg** $.n \Rightarrow$ **even** $.n$.

This didn't seem to tell him anything about blorgs, until he rewrote (2) equivalently, by taking the contrapositive:

for all n : $\neg$ **even** $.n \Rightarrow \neg$ **blorg** $.n$.

Now he could conclude that non-even (ie, odd) numbers were not blorgs. Of course, that can be read directly from (2) , which says that blorgs are even.

This definitely has something to do with the fact that **blorg** and **even** are boolean. For in set theory no one ever feels compelled to rewrite $S \subseteq T$ as $T^c \subseteq S^c$ in order to understand it better.

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Conclusion

In short, some of our interfaces with boolean concepts are rather unfortunate. We would be wise to find more productive ways of understanding them.

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