Skolemization and case analysis

How I understand Skolemization

Suppose we can calculate X (which doesn't depend on c) using property $p.c$. Then $[\ \langle\exists c :: p.c\rangle \Rightarrow X\ ]$. The proof is simple predicate calculus:

$$
\begin{array}{ll}
 & X \\
\Leftarrow & \{ \text{ predicate calculus } \} \\
 & \langle\exists c :: p.c\rangle \ \wedge\ X \\
\equiv & \{ \text{ predicate calculus, } X \text{ doesn't depend on } c \ \} \\
 & \langle\exists c ::\ p.c \ \wedge\ X\ \rangle \\
\equiv & \{ \text{ assumption } [\ p.c \Rightarrow X\ ]\ \} \\
 & \langle\exists c :: p.c\rangle \qquad .
\end{array}
$$

Naturally, $p.c$ may be a conjunction of properties of c used throughout the initial calculation of X .

Skolemization and case analysis

To my surprise, I recently discovered that calculational Skolemization is closely connected to case analysis.

Let me remind the reader of the technique of calculational case analysis, as described in JAW59 : A calculation can be made through a series of local calculations, provided the global context implies the disjunction of the local contexts.

The link between these techniques arose during the writing of JAW76 , where I was proving that if f is punctual, then it satisfies:

$$[\ f.(x \wedge y) \ \Leftarrow\ f.x \wedge f.y\ ] \qquad (\forall x, y) \qquad .$$

For my proof, I used a case analysis:

$$
\begin{array}{ll}
[\![& \text{Context: } \quad x \Rightarrow y \\
 & f.(x \wedge y) \\
\equiv & \{ \text{ context: } x \wedge y \equiv x \ , \quad f \text{ is punctual } \} \\
 & f.x \\
\Leftarrow & \{ \text{ predicate calculus } \} \\
 & f.x \ \wedge\ f.y \\
]\!] & \quad .
\end{array}
$$

The other case is $y \Rightarrow x$, and the proof is similar. The reader can check that the disjunction of the local contexts, $(x \Rightarrow y) \vee (y \Rightarrow x)$, indeed follows from the global context, in fact from predicate calculus.

Delighted by this proof, I attempted to generalize from finite conjunctivity to arbitrary conjunctivity, so my calculandum became:

$$[\ f.\langle\forall x :: x\rangle \ \Leftarrow\ \langle\forall x :: f.x\rangle\ ] \qquad .$$

The question before me was how to translate the case analysis. When we dealt with finite conjunctivity, the conjunction in question was $x \wedge y$, and there were two cases: one where this conjunction equivales x , and another where it equivales y . Working by analogy, now my conjunction is $\langle\forall x :: x\rangle$, I need one case for every x in the range of the universal quantification, and in each case, the conjunction will equivale the x chosen from the range.

This line of reasoning started to make me think of set theory. I got nauseated, and so just decided to calculate an "arbitrary" case:

$$
\begin{array}{ll}
 & f.\langle\forall x :: x\rangle \\
\equiv & \{ \text{ case: } \langle\forall x :: x\rangle \equiv x \ , \quad f \text{ is punctual } \} \\
 & f.x \\
\Leftarrow & \{ \text{ predicate calculus } \} \\
 & \langle\forall x :: f.x\rangle \qquad .
\end{array}
$$

The theory of case analyses now asks us to consider the disjunction of the local contexts, that is:

$$(0) \qquad \langle\exists x :: \ \langle\forall x :: x\rangle \equiv x \ \rangle \qquad .$$

The reader may check that (0) holds for nonempty ranges, hence the theorem in question may be generalized to universal quantifications over nonempty ranges.

But please observe that in the above calculation, the techniques of case analysis and Skolemization have completely coincided. Our calculandum:

$$f.\langle\forall x :: x\rangle \ \Leftarrow\ \langle\forall x :: f.x\rangle$$

is indeed independent of x , and was calculated with property $\langle\forall x :: x\rangle \equiv x$. Our obligation then, according to the technique of Skolemization, is to prove (0) .

* * *

Final thoughts

The calculational style has evolved over time, and for the better, I think. For instance, many calculations involving structures were rather ugly before the calculational world adopted J. T. Udding's suggestion to have an implicit pair of brackets around each line of a calculation. (See EWD835 .)

Thus it makes me very excited to see uses for these "exotic strands" of the calculational style, like Skolemization, structure-valued contexts, and perhaps even "Hehner's technique" , as discussed in JAW74 . For the record, I am firmly against complicating a simple system with all sorts of confusing bells and whistles; but at least in these cases, I think the use of these methods is justified by the sort of brevity and elegance thereby obtained.

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