For the record: A beautiful derivation

On page 67 of **[0]** , the authors point out that instead of postulating:

(0) $[\ \langle\forall x : [\,x = y\,] : f.x\rangle \ \equiv\ f.y\ ]$,

and then deriving the theorem:

(1) $\langle\forall x : [\,x = y\,] : \mathbf{false}\rangle \ \equiv\ \mathbf{false}$,

one could, in reverse, postulate (1) and derive (0) as a theorem. The authors "suggest the use of Leibniz's Rule" .

Not having much experience with serious manipulation of the predicate calculus, I couldn't resist taking the authors up on their challenge. Here is my proof, with heuristics:

$\langle\forall x : [\,x = y\,] : f.x\rangle$

$\equiv$ { In order to exploit (1) , we need to get **false** into the term. This can be done simply, using the fact that **false** is the unit of $\vee$. }

$\langle\forall x : [\,x = y\,] : f.x \vee \mathbf{false}\rangle$

$\equiv$ { Now that **false** has been introduced, we need to get rid of $f.x$, and since we also need to have $f.x$ conjoined with $[\,x = y\,]$ to use Leibniz's rule, shunting is called for. }

$\langle\forall x : [\,x = y\,] \wedge \neg f.x : \mathbf{false}\rangle$

$\equiv$ { As planned, we introduce $f.y$ of our goal, using Leibniz. }

$\langle\forall x : [\,x = y\,] \wedge \neg f.y : \mathbf{false}\rangle$

$\equiv$ { So far, so good! We are almost ready to use (1) , but first we have to remove $f.y$ from the quantification. This should be possible, since $f.y$ does not depend on x . Preparing to use the distributivity of $\vee$ over $\forall$, we trade $f.y$ back into the term. }

$\langle\forall x : [\,x = y\,] : f.y \vee \mathbf{false}\rangle$

$\equiv$ { As planned, we use the fact that $\vee$ distributes over $\forall$. }

$f.y \ \vee\ \langle\forall x : [\,x = y\,] : \mathbf{false}\rangle$

$\equiv$ { Now we are ready to use (1) ... }

$f.y \ \vee\ \mathbf{false}$

$\equiv$ { ... and **false** leaves as it entered. }

$f.y$.

This derivation is perhaps too simple to merit a JAW , which is why this is just "for the record" .

[0] Edsger W. Dijkstra & Carel S. Scholten: "Predicate Calculus and Program Semantics" , Springer-Verlag, New York Inc., 1990

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