The beauty of distributivity

Here is a familiar property of a pair of operators $*$ and $+$:

$$(0) \qquad x*(a+b) \;=\; (x*a)+(x*b) \qquad \text{—for all } x,a,b\text{—} \quad .$$

It is called: " $*$ distributes over $+$ (from the left)" , or simply "distributivity" . In my opinion, distributivity is the simplest interesting property of operators. Associativity and symmetry are important, but they lack interesting structure. Idempotence is too destructive. Probably monotonicity comes closest in terms of beauty in simplicity, but again it lacks the structure-changing and structure-preserving features that make distributivity so intriguing.

When we use (0) to rewrite one side as the other, certain structural relationships between the symbols change, while others remain the same. In this short note, I wish to show exactly how the use of distributivity affects the structure of a formula. I will do this by listing the structural properties of each side of (0) in a simple table.

First some terminology: For binary operator $\diamond$, I call an expression like $p\diamond q$ a $\diamond$-expression , with lefthand argument p and righthand argument q .

And now, without further ado, here is our table:

$x*(a+b)$	$(x*a)+(x*b)$
it is a $*$-expression	its arguments are $*$-expressions
its righthand argument is a $+$-expression	it is a $+$-expression
the lefthand argument of $*$ is x	each lefthand argument of $*$ is x
its righthand argument contains a and b	the righthand arguments are a and b
the lefthand argument of $+$ is a	the lefthand argument of $+$ contains a
the righthand argument of $+$ is b	the righthand argument of $+$ contains b

I invite the reader to take a moment and investigate this table, comparing the two sides. I feel that when the structural properties of distributivity are laid bare in this way, one can appreciate the beauty of distributivity in an entirely new way. For example, we can easily see that rewriting $(x*a)+(x*b)$ as $x*(a+b)$ allows us to join together a and b with $+$, whereas originally, the $*$-expressions they were *contained* in were joined by $+$. It also lets us rewrite a $+$-expression as a $*$-expression , or decrease the number of occurrences of x and $*$. And so forth.

Ultimately, however, words can only convey part of the beauty of the distributive property. Diethard Michaelis put it nicely: "Isn't the formula (like a picture) the beauty? It's the most compact way to do its beauty justice." .

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