Shrinking is strengthening, widening is weakening

This note is about a simple, effective heuristic for exploiting transitivity in the design of calculational arguments. It goes by the following catchphrase:

Shrinking is Strengthening
Widening is Weakening
.

To illustrate this heuristic, we consider a relation $\sqsubseteq$ satisfying the transitivity property, namely:

$$x \sqsubseteq y \ \wedge \ y \sqsubseteq z \quad \Rightarrow \quad x \sqsubseteq z \qquad \text{—for all } x, y, z \text{ —} \quad .$$

The terms **shrinking** and **widening** apply to the manipulation of expressions like $l \sqsubseteq r$. Roughly speaking, shrinking corresponds to moving l and r closer together, thereby shrinking the $(l..r)$ interval, while widening corresponds to moving them further apart, thereby widening that interval.

To explain further, focus on shrinking for a moment: What does it mean to "move l and r closer together" ? Notice that in $l \sqsubseteq r$, we have l on the left side, and r on the right. Thus, we move l and r closer together by moving l to the right, or by moving r to the left. But what does this mean? To move l to the right means to replace it with m , where we have $l \sqsubseteq m$. Similarly, to move r to the left means to replace it with q , where we have $q \sqsubseteq r$. Pretty simple! Hopefully the reader can now fully understand widening, by analogy.

Putting it all together: **Shrinking is Strengthening, Widening is Weakening** . Thus we may calculate as follows:

$$\begin{array}{ll} & l \sqsubseteq r \\ \Leftarrow & \{ \text{ shrinking, using } \ l \sqsubseteq m \ \} \\ & m \sqsubseteq r \qquad . \end{array}$$

(Given $l \sqsubseteq m$, we replace l with m , moving it to the right. This brings l and r closer together, thereby shrinking the $(l..r)$ interval. Shrinking is strengthening.)

This simple and powerful heuristic can help us in any calculation where we need to manipulate expressions involving transitive relations like $\leq$, $\Rightarrow$, $\subseteq$, the "divides" relation, and so forth.

A historical note: The term "widening" was first introduced by Wim Feijen and Netty van Gasteren in their book, *On a Method of Multiprogramming* . I came up with its counterpart, "shrinking" , and the catchphrase **Shrinking is Strengthening, Widening is Weakening** , during my year in Eindhoven.

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