Integer points in intervals

Nomenclature

In this note, x and y are real numbers, and $\#$ is the count operator, satisfying $\#\mathbf{true} = 1$ and $\#\mathbf{false} = 0$. We let $\mathcal{N}.x.y$ equal the number of integer points strictly between x and y . (Thus $\mathcal{N}$ is symmetric in its two arguments.) Finally, we let $n.i$ equal the number of integer points in the interval i .

Our theorem

We will derive conditions under which:

(0) $\quad \mathbf{even}.(\mathcal{N}.x.y) \;\equiv\; \mathbf{even}.(\mathcal{N}.(x+2).y) \quad .$

We distinguish two cases:

A: $\quad y \notin [\, x \,..\, x+2 \,]$

B: $\quad y \in [\, x \,..\, x+2 \,] \quad .$

Case **A**

We subdivide Case **A** into two further cases: $y < x$ and $x+2 < y$. We focus here on $x+2 < y$, since the other case follows from the symmetry of $\mathcal{N}$ in its arguments. We calculate:

$[\![$ Context: $\quad x+2 < y$

$\quad \mathcal{N}.x.y$

$= \quad \{ \; x+2 < y \;,\; \text{hence} \;\; x \le y \; \}$

$\quad n.(\, x \,..\, y \,)$

$= \quad \{ \; x \le x+2 \;\; \text{and} \;\; x+2 < y \; \}$

$\quad n.(\, x \,..\, x+2 \,] \;+\; n.(\, x+2 \,..\, y \,)$

$= \quad \{$ interval property on left term, $\; x+2 \le y \;$ on right term $\}$

$\quad 2 \;+\; \mathcal{N}.(x+2).y$

$]\!]$,

and hence we have:

$$\begin{array}{ll} & \mathbf{even}.(\mathcal{N}.x.y) \;\equiv\; \mathbf{even}.(\mathcal{N}.(x+2).y) \\ \equiv & \quad \{ \;\; \mathbf{even} \;\; \text{over} \;\; \equiv \;\; \} \\ & \mathbf{even}.(\; \mathcal{N}.x.y \; - \; \mathcal{N}.(x+2).y \;) \\ \equiv & \quad \{ \text{ above calculation } \} \\ & \mathbf{even}.2 \\ \equiv & \quad \{ \text{ property of } \;\mathbf{even}\; \} \\ & \mathbf{true} \qquad . \end{array}$$

Thus (0) holds in Case **A** with no additional assumptions.

Case **B**

In the second case, we have:

$[\![$ Context: $y \in [\, x \,..\, x+2 \,]$

$$\begin{array}{ll} & \mathcal{N}.x.(x+2) \\ = & \quad \{ \;\; x \le x+2 \;\; \} \\ & n.(\, x \,..\, x+2 \,) \\ = & \quad \{ \;\; y \in [\, x \,..\, x+2 \,] \;, \text{ see below for definition of } \; C \;\; \} \\ & n.(\, x \,..\, y \,) \; + \; C \; + \; n.(\, y \,..\, x+2 \,) \\ = & \quad \{ \;\; y \in [\, x \,..\, x+2 \,] \;\; \} \\ & \mathcal{N}.x.y \; + \; C \; + \; \mathcal{N}.(x+2).y \end{array}$$

$]\!]$,

where $C \; = \; \#(\, y \in \mathbb{Z} \cap (\, x \,..\, x+2 \,)\,)$. Thus we have:

$$\begin{array}{ll} & \mathbf{even}.(\mathcal{N}.x.y) \;\equiv\; \mathbf{even}.(\mathcal{N}.(x+2).y) \\ \equiv & \quad \{ \;\; \mathbf{even} \;\; \text{over} \;\; \equiv \;\; \} \\ & \mathbf{even}.(\; \mathcal{N}.x.y \; + \; \mathcal{N}.(x+2).y \;) \\ \equiv & \quad \{ \text{ above calculation } \} \\ & \mathbf{even}.(\; \mathcal{N}.x.(x+2) \; - \; C \;) \\ \equiv & \quad \{ \;\; \mathbf{even} \;\; \text{over} \;\; - \;\; \} \\ & \mathbf{even}.(\mathcal{N}.x.(x+2)) \;\equiv\; \mathbf{even}.C \end{array}$$

$\equiv$ { interval property on left term, definition of C on right term }

$\neg(x \in \mathbb{Z}) \equiv \neg(y \in \mathbb{Z} \cap (x..x+2))$

$\equiv$ { predicate calculus }

$x \in \mathbb{Z} \equiv y \in \mathbb{Z} \cap (x..x+2)$.

Thus (0) holds precisely when either:

P: $x \in \mathbb{Z} \wedge y \in \mathbb{Z} \cap (x..x+2)$ or

Q: $x \notin \mathbb{Z} \wedge y \notin \mathbb{Z} \cap (x..x+2)$.

As the reader can verify, **P** and **Q** can be written equivalently as:

P: $x \in \mathbb{Z} \wedge y = x+1$ and

Q: $x \notin \mathbb{Z} \wedge y \notin \mathbb{Z}$.

<u>Putting it together</u>

Putting Case **A** and Case **B** together, we see that (0) equivales:

$(y \notin [\![x..x+2]\!] \wedge \textbf{true}) \vee (y \in [\![x..x+2]\!] \wedge (\mathbf{P} \vee \mathbf{Q}))$

$\equiv$ { predicate calculus }

$y \notin [\![x..x+2]\!] \vee \mathbf{P} \vee \mathbf{Q}$

$\equiv$ { **P** and **Q** }

$y \notin [\![x..x+2]\!] \vee (x \in \mathbb{Z} \wedge y = x+1) \vee (x \notin \mathbb{Z} \wedge y \notin \mathbb{Z})$.

In words, (0) holds precisely when either:

- y is outside the interval $[\![x..x+2]\!]$, or
- x and y are both integers, with $y = x + 1$, or
- x and y are both nonintegers .

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