On humble nodes and not-so-humble mathematicians

Recently, my colleague Chris Phelps turned me on to a blog devoted to mathematical puzzles.[1] I was intrigued by a graph theory problem which seemed quite simple at first, but which turned out to contain more than a few surprises. After a week and a half of pondering (with some help from Tom Verhoeff —in the flesh!— and Jake Boggan), I felt ready to record my thoughts in a note.

First, some terminology: A graph consists of a collection of nodes, each pair of which may or may not be joined by an edge. Two nodes joined by an edge are said to be adjacent, and the nodes adjacent to a given node are called its neighbors. The degree of a node is the number of neighbors it has.

Now for the problem: Let us call a node humble if its degree is at most the average of the degrees of its neighbors.[2] Given an arbitrary graph with at least 2 nodes, what is the greatest number of humble nodes the graph is guaranteed to contain?

The answer is 2. That we cannot guarantee more than 2 humble nodes can be seen from the following construction: Consider an arbitrary graph with at least 2 nodes. Join each pair of nodes with an edge, then remove 1 edge arbitrarily. The resulting graph has precisely 2 humble nodes, as the following proof shows.

Proof. Let us say that the graph has N nodes and that the removed edge joined nodes x and y. Before the edge is removed, each node has degree $N-1$ (because each node is joined to the other $N-1$ nodes). Thus, after the edge is removed, x and y have degree $N-2$, while each other node still has degree $N-1$. These other nodes are not humble because they are adjacent to x and y, and hence the average of the degrees of their neighbors is strictly less than their degree, $N-1$. On the other hand, x and y are humble because all their neighbors have greater degree. (End of Proof.)

This construction can be considered ‘easy’, in that it was discovered by myself, by all my colleagues, and by all the commenters on the blog. That we can guarantee at least 1 humble node in an arbitrary graph is also ‘easy’: consider a node of minimal degree. That

[1] http://www.fivethirtyeight.com/tag/the-riddler/

[2] The original problem was given in terms of the opposite concept, ‘proud’.

we can guarantee 2 humble nodes in every graph was deemed obvious, but I was the first to furnish a correct (not easy!) proof:

Proof in three cases.

Consider an arbitrary graph with at least 2 nodes. Let P be a node of minimal degree. Then P is humble because all of its neighbors have equal or greater degree. If another node has the same degree as P, it is humble for the same reason. (Case 0)

If there is no such node, then consider the collection of nodes with second smallest degree — we call these nodes <u>second-placers</u> . If a second-placer is not adjacent to P, it is humble because all of its neighbors have equal or greater degree. (Case 1)

Finally, if all second-placers are adjacent to P, then —as we will see— each second-placer is humble. (Case 2)

Let s be the number of second-placers and let the degree of a second-placer exceed the degree of P by k. Then the degree of P is at least s (because P is adjacent to all the second-placers) , and the degree of a second-placer is at least $s + k$.

Each second-placer has at most s neighbors with equal or lesser degree —namely P and the other $s - 1$ second-placers— , hence each second-placer has at least k neighbors with degree at least 1 greater. But each second-placer has only 1 neighbor of lesser degree: namely P, which has degree k less. Hence the average of the degrees of a second-placer's neighbors is at least the degree of that second-placer; in other words, each second-placer is humble.

(End of Proof.)

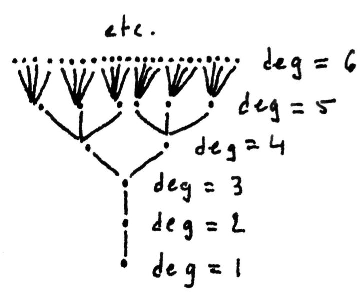
At first, I considered this proof to be Very Ugly, because the existence of one humble node is proved very easily, while the existence of a second humble node is considerably more involved. It seemed obvious that the existence of a second humble node should be Very Easy to prove. Indeed, many commenters on the blog proposed simpler proofs along different lines — none of which, however, were correct!

In the above proof, P is 'trivially' humble, in the sense that none of its neighbors have smaller degree, hence P's degree is at most the average of its neighbors' degrees. Contrast this with the final part of my proof, where I showed that the second-placers adjacent to P are humble: these nodes have neighbors both of smaller and greater degree, and we had to carefully count and average to conclude humility. (How fitting.)

It seemed clear to me and the commenters that this fanfare about second-placers was unnecessary: surely all graphs contain another node like P.

To explore this notion, let us give a name to 'what P is like' : Let us say that a node is a <u>root degree node</u> if none of its neighbors have smaller degree. Nodes of minimal degree are root degree nodes, and all root degree nodes are humble. What I and the other commenters tacitly assumed is that every graph has not only 2 humble nodes, but in fact 2 root degree nodes. Again, this seemed obvious — but where was the proof?

The first sign that there was something more subtle going on came when I realized that it was possible for infinite graphs (graphs with an infinite collection of nodes but where each node has finite degree) to have only 1 root degree node. For example:

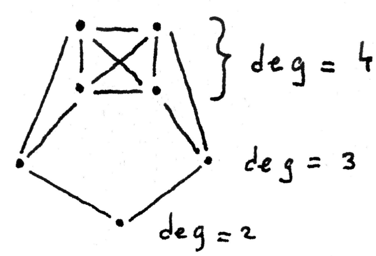


Each node of this graph other than the bottom-most node has neighbors both of smaller and greater degree.

But consider that my proof that any graph contains 2 humble nodes holds equally well for infinite graphs. (Indeed, in the above graph, all nodes are humble!) Hence any proof demonstrating the existence of 2 humble nodes by showing the existence of 2 root degree nodes would have to take the finiteness of the graph into account somehow. None of the commenters' proofs had done so, which is why they were doomed to fail.

This proof strategy only works, of course, if finite graphs *have* at least 2 root degree nodes, but this seemed clear — the shenanigans in the above illustration wouldn't be remotely possible in a finite graph.

I should have been more humble. After all, the efforts of several fine mathematicians hadn't yet found a second root degree node! Indeed, using the definition, I designed the following minimal counterexample to the conjecture that a finite graph has at least 2 root degree nodes:



Only the bottom-most node is a root degree node. (Note nevertheless that the 'second-placer' nodes of degree 3 are humble, as my proof requires.)

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The experience of wrestling with these concepts was, in a word, humbling! I still think a nicer proof of the existence of 2 humble nodes may exist, but I no longer feel like my proof missed anything obvious.

Indeed, generally speaking, some things that seem obvious are; others aren't. And it's not always obvious which are which. A curious, questioning, mathematical mindset humbles us, and helps us to see the difference.

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