Induction without a base step

From Canada/USA Mathcamp's 2016 qualifying quiz comes the following problem:

> *We define a property* <u>*oddly nice*</u> *on positive integers as follows:*
>
> - 1 *is oddly nice*
> - $n > 1$ *is oddly nice iff an odd number of its proper divisors are oddly nice.*
>
> *Which positive integers are oddly nice? Let* $s(n)$ *denote the number of oddly nice proper divisors of* n *. What are the possible values of* $s(n)$ *?*

Without making any effort at deriving the following solution, I present it simply because it has two nice uses of mathematical induction, one without a base step.

The positive integers come in two kinds: *skinny* numbers, which are products of distinct primes; and *fat* numbers, which are not, and therefore have a repeated prime factor. I claim that the oddly nice numbers are precisely the skinny ones.

Proof.

We show firstly that all skinny numbers are oddly nice, and secondly that all fat numbers are not.

Consider a skinny number which is the product of N distinct primes. Then it has 2^N divisors, $2^N - 1$ of which are proper and hence, by induction, oddly nice. So the skinny number in question is oddly nice if $2^N - 1$ is odd, which it is provided $N \neq 0$. When $N = 0$, the number in question is 1 , which is oddly nice by stipulation. So in any case, a skinny number is oddly nice.

Next consider a fat number with N distinct prime factors. Then it has 2^N skinny divisors, all of which are proper (since they aren't fat) and hence, by induction, oddly nice. The rest of the proper divisors are fat, and hence, by induction, not oddly nice. So the fat number in question is not oddly nice if 2^N is even, which it is since fat numbers have a repeated prime factor and hence $N \geq 1$.

(End of Proof.)

Note that at least one of the two induction proofs was bound to involve a case analysis, because —as we learned in the first induction proof— the two cases in the definition of 'oddly nice' cannot be wrapped into one.

We also see from the above proof that the answer to the second part of the question is that the possible values of s are $2^N - 1$ for any $N \geq 0$, and 2^N for any $N \geq 1$.

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