The rational square roots are the perfect squares

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In the late summer of 1992, I showed Greg Nelson my proof of the well-known property that $\sqrt{2}$ is irrational [KRML 0]. Greg then mentioned to me that a positive integer has a rational square root exactly when it is a perfect square, a property at that time unknown to me. We saw how to extend my proof to show the stronger property, but did not record the proof, let alone write it down.

Recently, the notation in [RVM 5] inspired me to write positive numbers as tagged collections of natural numbers, the index set being the set of primes. This reminded me of the proof alluded to above. Reconstructing the argument, I found, however, that I could write it comfortably without the notation from [RVM 5]. Unconventionally deciding to be conventional, I left the notation at that.

Enter, Greg and my proof.

For any prime p and positive integer n, we let $P.p.n$ denote the number of occurrences of p in a prime factorization of n. For any such n, we calculate,

$$
\begin{array}{ll}
 & \sqrt{n} \text{ is rational} \\
= & \qquad \{ \quad \text{def. rational} \quad \} \\
 & \langle \exists x, y \mid y \neq 0 \triangleright \sqrt{n} = x/y \rangle \\
= & \qquad \{ \quad \sqrt{n} > 0 \text{, and math} \quad \} \\
 & \langle \exists x, y \mid x > 0 \wedge y > 0 \triangleright \sqrt{n} = x/y \rangle \\
= & \qquad \{ \quad \text{math} \quad \} \\
 & \langle \exists x, y \mid x > 0 \wedge y > 0 \triangleright n \cdot y^2 = x^2 \rangle \\
= & \qquad \{ \quad \text{prime factorization} \quad \} \\
 & \langle \exists x, y \mid x > 0 \wedge y > 0 \triangleright \langle \forall p \triangleright P.p.(n \cdot y^2) = P.p.(x^2) \rangle \rangle \\
= & \qquad \{ \quad P.p.(a \cdot b) = P.p.a + P.p.b \quad \} \\
 & \langle \exists x, y \mid x > 0 \wedge y > 0 \triangleright \langle \forall p \triangleright P.p.n + 2 \cdot P.p.y = 2 \cdot P.p.x \rangle \rangle \\
\Rightarrow & \qquad \{ \quad a + b = c \wedge b \text{ even} \wedge c \text{ even} \Rightarrow a \text{ even} \quad \} \\
 & \langle \exists x, y \mid x > 0 \wedge y > 0 \triangleright \langle \forall p \triangleright P.p.n \text{ even} \rangle \rangle \\
= & \qquad \{ \quad \text{range of } x, y \text{ nonempty} \quad \} \\
 & \langle \forall p \triangleright P.p.n \text{ even} \rangle \\
= & \qquad \{ \quad \text{def. perfect square} \quad \} \\
 & n \text{ is a perfect square} \qquad .
\end{array}
$$

By definition, a perfect square has an integral, and thus rational, square root. Hence, we have established, for all positive n,

$$\sqrt{n} \text{ rational} \equiv n \text{ is a perfect square} \qquad .$$

References

[KRML 0] K.R.M. Leino. $\sqrt{2}$ *is irrational, revisited*, 1991.

[RVM 5] K.R.M. Leino and Rajit Manohar. *Lists as quantifiers*, 1994.