

Settling Multiple Debts Efficiently: Problems

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Abstract

I present a series of problems related to [1] for use in the class room.

1 Introduction

A group of friends lend each other money throughout the year. They carefully record each transaction. When Alice lends 10 euro to Bob, this is recorded as Alice $\xrightarrow{10}$ Bob.

At the end of the year they wish to settle their debts. How should they transfer money so as to settle all debts? Of course, they could reverse the action of each recorded loan separately. However, often there is a better way.

The problems posed below concern efficient ways of settling all debts. Try to minimize the number of transfers and the total amount transferred. We consider only settling schemes, where money is transferred between a pair of persons.

2 Problems

1. After the loans

$$\begin{array}{l} \text{Alice} \xrightarrow{10} \text{Bob} \\ \text{Carol} \xrightarrow{10} \text{Dick} \end{array}$$

is this a proper settlement:

$$\begin{array}{l} \text{Bob} \xrightarrow{10} \text{Carol} \\ \text{Dick} \xrightarrow{10} \text{Alice} \end{array}$$

(involving 2 transfers for a total amount of 20 euro)?

2. How to settle efficiently the loans:

$$\begin{array}{l} \text{Alice} \xrightarrow{10} \text{Bob} \\ \text{Alice} \xrightarrow{10} \text{Carol} \\ \text{Bob} \xrightarrow{10} \text{Carol} \end{array}$$

3. How to settle efficiently the loans:

$$\begin{array}{l} \text{Alice} \xrightarrow{10} \text{Bob} \\ \text{Alice} \xrightarrow{20} \text{Dick} \\ \text{Carol} \xrightarrow{30} \text{Bob} \\ \text{Carol} \xrightarrow{40} \text{Dick} \end{array}$$

4. How to settle efficiently the loans:

$$\begin{array}{l} \text{Alice} \xrightarrow{20} \text{Bob} \\ \text{Bob} \xrightarrow{10} \text{Carol} \\ \text{Alice} \xrightarrow{60} \text{Dick} \\ \text{Dick} \xrightarrow{30} \text{Carol} \\ \text{Carol} \xrightarrow{20} \text{Alice} \end{array}$$

5. How can the settlement

$$\begin{array}{l} \text{Alice} \xrightarrow{10} \text{Carol} \\ \text{Carol} \xrightarrow{60} \text{Dick} \\ \text{Dick} \xrightarrow{20} \text{Bob} \end{array}$$

be improved to minimize the total amount transferred?

6. How can the settlement

$$\begin{array}{l} \text{Alice} \xrightarrow{20} \text{Bob} \\ \text{Alice} \xrightarrow{30} \text{Dick} \\ \text{Carol} \xrightarrow{30} \text{Bob} \end{array}$$

be improved to minimize the number of transfers?

7. Describe a general scheme for settling all debts among N persons in at most $N - 1$ transfers and with a minimal total amount transferred.

3 Solutions

1. Yes, although that may seem counterintuitive at first.

Of importance is the *balance* of each person: the total amount lend to others minus the total amount borrowed from others.

Alice and Carol have a credit of 10 euro (balance +10), whereas Bob and Dick have a debt of 10 euro (balance -10). If the friends don't mind what money they get from whom, but only care about the amount, then the proposed settlement is equivalent to the more obvious settlement:

$$\begin{array}{l} \text{Bob} \xrightarrow{10} \text{Alice} \\ \text{Dick} \xrightarrow{10} \text{Carol} \end{array}$$

It has the same number of transfers and the same total amount transferred.

2. The balances are:

Ada	+20
Bob	0
Carol	-20

Hence an 'optimal' settlement is

$$\text{Carol} \xrightarrow{20} \text{Alice}$$

with a single transfer of 20 euro.

3. The balances are:

Alice	+30
Bob	-40
Carol	+70
Dick	-60

Hence an 'optimal' settlement is

$$\begin{aligned} \text{Bob} &\xrightarrow{30} \text{Alice} \\ \text{Bob} &\xrightarrow{10} \text{Carol} \\ \text{Dick} &\xrightarrow{60} \text{Carol} \end{aligned}$$

Also 'optimal' is

$$\begin{aligned} \text{Bob} &\xrightarrow{40} \text{Carol} \\ \text{Dick} &\xrightarrow{30} \text{Alice} \\ \text{Dick} &\xrightarrow{30} \text{Carol} \end{aligned}$$

Both involve three transfers for a total amount of 100 euro.

4. The balances are:

Alice	+60
Bob	-10
Carol	-20
Dick	-30

Hence an 'optimal' settlement is

$$\begin{aligned} \text{Bob} &\xrightarrow{10} \text{Alice} \\ \text{Carol} &\xrightarrow{20} \text{Alice} \\ \text{Dick} &\xrightarrow{30} \text{Alice} \end{aligned}$$

with three transfers for a total amount of 60 euro.

5. The balances before the settlement were:

Alice	-10
Bob	+30
Carol	-50
Dick	+30

Hence an 'optimal' settlement is for example

Alice	$\xrightarrow{10}$	Bob
Carol	$\xrightarrow{20}$	Bob
Carol	$\xrightarrow{30}$	Dick

with three transfers for a total amount of only 60 euro (instead of 90). Bob and Dick can also be exchanged.

Note that the settlement

Alice	$\xrightarrow{10}$	Carol
Carol	$\xrightarrow{30}$	Bob
Carol	$\xrightarrow{30}$	Dick

is an improvement with a total transferred amount of 70 euro, but it is not 'optimal'.

6. The balances before the settlement were:

Alice	-50
Bob	+50
Carol	-30
Dick	+30

Hence an 'optimal' settlement is

Alice	$\xrightarrow{50}$	Bob
Carol	$\xrightarrow{30}$	Dick

with only two transfers (instead of three) for a total amount of 80 euro.

7. **Step 1.** Determine the balance for each person.

Observe that the sum of all balances equals zero.

Let P be the total amount of positive balances, and N the total amount of negative balances. Hence, $P = -N$.

The minimum total amount to be transferred equals P .

Step 2. While there is still someone with a nonzero balance, do:

Step 2a. Select a person A with a negative balance $S < 0$, and a person B with a positive balance $T > 0$ (these exist).

Step 2b. Let M be the minimum of $-S$ and T . Hence, $M > 0$.

Step 2c. Include the transfer $A \xrightarrow{M} B$ in the settlement.

Step 2d. Increase the balance of A by M and decrease the balance of B by M (the total balance remains zero).

Observe that at least one of A and B now has a balance zero.

Step 3. All balances are zero, hence the included transfers settle all debts.

The total amount transferred equals P , and hence is minimal. The repetition of Step 2 terminates, because in each iteration at least one nonzero balance is reduced to zero. Therefore, the number of transfers is at most N . In fact, it is at most $N - 1$, because the final two nonzero balances cancel each other in a single transfer.

The settlement obtained by this algorithm is not guaranteed to have a minimal number of transfers. For example, the nonoptimal settlement of Problem 6 can be obtained from the balances via the algorithm above.

It is not known how to minimize the number of transfers efficiently. This is a NP-hard problem, for which only algorithms are known that “try” an exponential number of possibilities.

References

- [1] Tom VERHOEFF. “Settling Multiple Debts Efficiently: An Invitation to Computing Science”, *Informatics in Education*, **3**(1):105–126 (2004). Dover Publications.