The Boyer - Moore Majority Vote (for our records)

For any bag B of integers the equation
$m: \; (\underline{N}x: x \; \underline{in} \; B: x = m) - (\underline{N}x: x \; \underline{in} \; B: x \neq m) \geq 1$
has at most one solution. If it has a solution, that solution is the bag's majority.

We observe

Lemma 0 A bag B for which the equation
$m: \; (\underline{N}x: x \; \underline{in} \; B: x = m) - (\underline{N}x: x \; \underline{in} \; B: x \neq m) = 0$
has a solution, is without majority.
(End of Lemma 0.)

Lemma 1 The union of two bags without majority is without majority.
(End of Lemma 1.)

Lemma 2 At least one of the subbags in a partitioning of a bag with majority m has majority m.
(End of Lemma 2.)

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The above lemmata serve to argue about a program examining whether or not a given sequence $X(i: 0 \leq i < N)$, $N \geq 1$, has a majority.

$h := 0;\; n := 1;\; m := X(0);\; d := 1$
{invariant $P0 \wedge P1 \wedge P2$,
  $P0: \; 0 \leq h < n \leq N$
  $P1: \; X(i: 0 \leq i < h)$ is without majority
  $P2: \; 0 \leq d \wedge d = (\underline{N}i: h \leq i < n: X(i) = m)$
          $- (\underline{N}i: h \leq i < n: X(i) \neq m)$
}
; $\underline{do}$ $n \neq N$
  $\rightarrow$ $\underline{if}$ $X(n) = m \rightarrow d := d + 1$

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   [] X(n) ≠ m
      → if d ≥ 1 → d := d - 1
        [] d = 0 → { X(i: h ≤ i < n) is without majority
                      (P2 and Lemma 0), hence X(i: 0 ≤ i < n)
                      is without majority (P1 and Lemma 1) }
                      h := n {P1} ; m := X(n); d := 1 {P2^n_(n+1)}
        fi
   fi
  ; n := n + 1
od
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Upon completion of the program we conclude for a sequence $X(i: 0 \leq i < N)$ with majority that (P_1 and Lemma 1) $X(i: h \leq i < N)$ has a majority, which is (P_2 and Lemma 0) m, which is (P_1 and Lemma 2) the majority of $X(i: 0 \leq i < N)$.

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Remarks

a) The variable h is an auxiliary variable that can be eliminated from the program text.

b) P_2 is the part of the invariant from which a stepwise development of the program may start.

c) The above algorithm has been found by Robert S. Boyer and J. Strother Moore, a contribution for which they are acknowledged.

d) A.B.J. Kuylaars, a first year student in mathematics, came with the above program within at most half an hour, of which he can be proud.

(End of Remarks)

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