On the construction of the regular pentagon

The purpose of this note is to present heuristic considerations that lead to the construction, with ruler and compass alone, of the regular pentagon. Our effort was inspired by the complete absence of such considerations from, for instance, "What is Mathematics?" by Courant and Robbins, and "Introduction to Geometry" by Coxeter.

Courant and Robbins's presentation strongly suggests that it is most convenient to construct the regular decagon first. We were a little shocked by this suggestion, because we had no idea how a systematic development of the construction could exorcise that rabbit-pulled-out-of a hat. (Why, for instance, shouldn't we start with the regular icosagon?) Coxeter only enumerates the movements to be performed with ruler and compass, without any further explanation or justification.

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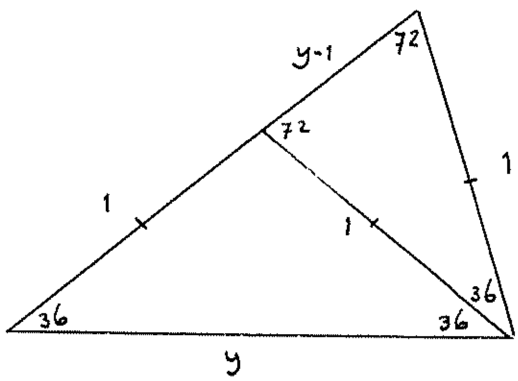
The problem is to construct 5 points evenly distributed on a circle, or to "pentasect" 360°. Exploiting the 5-fold symmetry, we decide to do so by constructing an angle of 72°. Because triangles are about the simplest figures that contain angles, we decide to do so by constructing a triangle containing an angle of 72°.

Simplicity suggests that of all such triangles we choose one with as few degrees of freedom as possible. An equilateral triangle has the fewest degrees of freedom, but it does not have an angle of 72°. The only next-simplest triangles we can think of are isosceles triangles and triangles with a simply constructable angle, such as 90° or 60° : their only degree of freedom is the ratio of two of their sides. This leaves us with the choice among

72°, 72°, 36°	72°, 90°, 18°
72°, 54°, 54°	72°, 60°, 48°
	72°, 45°, 63°
	72°, 30°, 78° ...

Of these, the first isosceles one -- 72°, 72°, 36° -- offers the simplest opportunity to formulate an equation for the ratio of sides: because $72 = 2 \times 36$, the interior angular bisector of an angle of 72° partitions the triangle into two isosceles triangles, viz. with the original triangle having base 1 and sides y we get



By similarity of triangles, (or by some theorem about bisectors,) we have $y : 1 = 1 : y-1$, i.e. $y^2 - y - 1 = 0$, i.e. $y = (1+\sqrt{5})/2$. Since $\sqrt{5}$ is constructable, so is $(1+\sqrt{5})/2$, hence so is the above triangle with angles 72° and 36°.

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We now have exorcised Courant and Robbins's rabbit. By "coincidence", the triangle we were most interested in contains the main ingredient for the regular decagon, the angle of 36°, as well. We did not have to mention decagons, however, nor do we have to perform the more elaborate construction of the regular decagon first.

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