An exercise of Richard Bird's

The following theorem was communicated to the world by Richard S. Bird.

Theorem

For $f: Nat \rightarrow Nat$ so that

(0) $(\underline{A}n: 0 \le n: \ f^2.n < f.(n+1))$,

we have $(\underline{A}n: 0 \le n: \ f.n = n)$

(End of Theorem.)

In this note we only record the facts that constitute our proof, leaving the heuristic considerations for later.

We prove the theorem by proving $f.n \le n$ and $n \le f.n$ separately

Lemma 0 For all natural n, $f.n \le n$

Proof For any n, $0 \le n$, we observe

$f.n \le n$
= { $f.n$ and n are integers }
$f.n < n+1$
= { Lemma 1 below, using
  (f is increasing)
    $\equiv (\underline{A}i,j:: i<j \equiv f.i < f.j)$
  }
$f.(f.n) < f.(n+1)$
= { datum (0) }
true .

(End of Proof.)

<u>Lemma 1</u> f is increasing.

<u>Proof</u> For any n, $0 \le n$, we observe

$$\begin{array}{ll} & f.(n+1) \\ > & \quad \{\text{datum } (0)\} \\ & f.(f.n) \\ \ge & \quad \{\text{Lemma 2 below}\} \\ & f.n \end{array}$$

(<u>End</u> of Proof.)

<u>Lemma 2</u> For all natural n, $n \le f.n$
(which is the other macroscopic conjunct.)

<u>Proof</u> The demonstrandum follows from

$H.n$: $(\underline{A}k : 0 \le k : n \le k \Rightarrow n \le f.k)$,

by instantiating it for $k := n$. We show

$$(\underline{A}n : 0 \le n : H.n)$$

by mathematical induction.

The base $n = 0$ follows from f's naturalness.

For $n+1 \le k$, which implies $1 \le k$, we observe

$$\begin{array}{ll} & n+1 \le f.k \\ = & \quad \{f.k \text{ and } n \text{ are integers}\} \\ & n < f.k \\ \Leftarrow & \quad \{\text{datum } (0),\ \text{using } 1 \le k\} \\ & n \le f.(f.(k-1)) \\ \Leftarrow & \quad \{H.n \text{ with } k := f.(k-1)\} \\ & n \le f.(k-1) \end{array}$$

$\Leftarrow \quad \{ H.n \text{ with } k := k-1 \}$

$\quad n \leq k-1$

$= \quad \{ \text{since } n+1 \leq k \}$

$\quad true$.

(*End* of Proof.)

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PS. The above calculations could be done by heart.