The "first closure lemma" in practice: two very small examples

We consider a universe with a (complete) partial order $\leq$. By definition, function f on that universe is a closure whenever

f is expanding , i.e. $x \leq f.x$ $(\forall x)$,
and f is monotonic with respect to $\leq$,
and f is idempotent

Now, Rutger M. Dijkstra revealed to us — in his master's thesis — what he dubbed "The First Closure Lemma":

$$
\begin{array}{l}
f \text{ is a closure} \\
\equiv \\
\langle \forall x,y :: \quad f.x \leq f.y \;\equiv\; x \leq f.y \rangle
\end{array}
$$

(, not proven here). This is a very nice way to capture at one fell swoop the three different characteristics "expanding", "monotonic", and "idempotent".

The purpose of this note is to record two very small examples illustrating the very effectiveness of the closure lemma.

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Chritiene Aarts, in his master's thesis, gives a very nice definition of the function known as the "ceiling" in real arithmatic. The ceiling, denoted $\lceil . \rceil$, is defined by

$$\lceil . \rceil \ : \ Int \leftarrow Real \quad ,$$

and $\langle \forall x,y : Real.x \wedge Int.y : \ \lceil x \rceil \leq y \ \equiv \ x \leq y \rangle$.

Now we show that $\lceil . \rceil$ is a closure. Using the closure lemma, we then have to show

$$\langle \forall x,y : Real.x \wedge Real.y : \ \lceil x \rceil \leq \lceil y \rceil \ \equiv \ x \leq \lceil y \rceil \rangle ,$$

and this follows immediately from $\lceil . \rceil$'s definition with the instantiation $x,y := x, \lceil y \rceil$. (Observe that we use the type information of $\lceil . \rceil$.)

Isn't this nice? (I have no idea how traditional mathematics handles the above, but I think that we can only shudder at the thought.)

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Next example. The so-called transitive closure of a relation r — denoted Ψr — can be defined as the smallest solution of the equation $x: \ r + x;x \leq x$. (Operators $+$ and $;$ are the usual operators from the regularity calculus, and we don't bother to explain them here.) Spelled out, Ψr is defined by

$$(0) \qquad r + x;x \leq x \ \Rightarrow \ \Psi r \leq x \qquad (\forall x)$$

— Ψr's extremity —

$$(1a) \qquad r \le \dagger r$$
$$(1b) \qquad \dagger r ; \dagger r \le \dagger r$$

— the "solves"-part, spelled out —

In WF175 (almost finished, and to be finished soon) we showed that $\dagger$ is a closure by explicitly showing that $\dagger$ is expanding, monotonic, and idempotent. One can imagine that these three proofs together consume quite some space. Now, let us use the closure lemma. For $\dagger$ to be a closure we have to prove

$$\dagger r \le \dagger s \quad \equiv \quad r \le \dagger s \qquad (\forall r, s) \quad .$$

Here we go:

$$
\begin{array}{ll}
\quad \dagger r \le \dagger s & (*) \\
\Leftarrow \quad \{ \text{extremity of } \dagger r \text{, see } (0) \} & \\
\quad r + \dagger s ; \dagger s \le \dagger s & \\
\equiv \quad \{ \text{reg. calc} \} & \\
\quad r \le \dagger s \;\wedge\; \dagger s ; \dagger s \le \dagger s & \\
\equiv \quad \{ (1b) \text{ with } r := s \} & \\
\quad r \le \dagger s & (**) \\
\Leftarrow \quad \{ r \le \dagger r \text{ , see } (1a) \} & \\
\quad \dagger r \le \dagger s \;, & (***)
\end{array}
$$

and the result follows from the marked lines.

Isn't this nice? I think it is. One up for the closure lemma. And I am convinced that there will be many, many more applications to emerge in the future.

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Jaap van der Woude pointed out to me that the closure lemma also occurs in the red book of R.C. Backhouse et.al., and that it is not new at all and that it is well-known for many decades in many branches of mathematics. True as all this may be, should I now apologize and should I have refrained from writing this note? My answer is a twofold no!, and for at least three reasons.

First – and this is quite personal – if I don't write it down, I am going to forget it. (Furthermore, I enjoy writing down what I consider nice mathematics.)

Second, I completely understand why this particular lemma in the red book did not catch my eyes. Not only is it presented in a generalized and "lifted" form – which is not too nice for a first exposure – but, worse, it occurs as one formula in a huge train of over thousand formulae, and as such has no outstanding position. In this respect the red book joins the major part of the mathematical texts which offer an infinite mass of more or less related results. My head is just too small to master all that.
(I am very well aware that one particular theory can host an abundance of theorems, but that only a small fraction of them forms the heart of the theory, as far as its contents and methodological issues are concerned. Any decent presentation of a theory has to emphatically highlight that heart.)

Third, I know that I have had a defective mathematical education (and that I am not the only one). From a computing science perspective we have learned that a streamlining of the mathematical argument is not just necessary but jolly well possible as well. We have to work on that mission and record what we consider worthwhile, lest the next generation will be exposed to mathematical curriculae as cumbersome as ours.

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