Two equivalent ceilings

(This note is written to convey to David Gries and Fred Schneider a potential exercise to be used in their courses on elementary, calculational mathematics. It is written for my own records also.)

We deem function $\lceil\cdot\rceil$ to be defined by:

For each x : Real.x

$A0$: $\quad$ Int. $\lceil x \rceil$

$A1$: $\quad \langle \forall y : \text{Int}.y : \lceil x \rceil \leq y \equiv x \leq y \rangle$

(Those who are familiar with the theory of Galois-Connexions will immediately see that $\lceil\cdot\rceil$ has thus been introduced as — what the jargon calls — a "lower Galois adjoint". But for our purposes we do not need to know this.)

Also, we deem defined function $\Uparrow$ by:

For each x : Real.x,

H: $\quad \Uparrow x = \langle \Downarrow y : \text{Int}.y \wedge x \leq y : y \rangle$.

(This is the traditional definition of the ceiling, which we have temporarily denoted by $\Uparrow$.)

Our purpose is to prove that $\lceil\cdot\rceil = \Uparrow$.

i.e. for each x: Real.x ,

Z: $\lceil x \rceil = {\Uparrow}x$.

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For the sake of proving Z we first spell out ⇑'s definition as given by H :

For each x: Real.x

B0: Int.$({\Uparrow}x)$

B1: $x \leq {\Uparrow}x$

B2: $\langle \forall y : \text{Int}.y : x \leq y \Rightarrow {\Uparrow}x \leq y \rangle$

(B0 and B1 express that ⇑x belongs to y's range in H , and B2 expresses the minimality of ⇑x .)

Now, we take the givens A and B as our point of departure in proving Z .

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Because in A and B the symbol = is absent and the symbols ≤ abound, we rewrite Z — using the antisymmetry of ≤ — into

$$\lceil x \rceil \leq {\Uparrow}x \quad \wedge \quad {\Uparrow}x \leq \lceil x \rceil \quad ,$$

and deal with these conjuncts separately .

(As usual, each of these conjuncts is tackled by setting up for it a strengthening calculational chain that ends with true .)

- $\lceil x \rceil \leq \Uparrow x$

$\Leftarrow$ { (A1 $\Leftarrow$) with $y := \Uparrow x$; this instantiation for y requires Int.y, but this exactly the given B0 }

$x \leq \Uparrow x$

$\equiv$ { B1 }

true .

<u>Remark</u> The first step of the above calculation only uses the $\Leftarrow$-part of A1, and that suffices for a strengthening chain. We intentionally refrained from using the fully-fledged A1, because the above setting makes more clear that we have not used (A1 $\Rightarrow$) yet. And this can be of importance for the heuristic of the design.)
(<u>End</u> of Remark.)

- $\Uparrow x \leq \lceil x \rceil$

$\Leftarrow$ { B2 with $y := \lceil x \rceil$. This instantiation for y is allowed because of A0 }

$x \leq \lceil x \rceil$

$\Leftarrow$ { (A1 $\Rightarrow$) with $y := \lceil x \rceil$, using A0 }

$\lceil x \rceil \leq \lceil x \rceil$

$\equiv$ { $\leq$ is reflexive }

true .

Final technical remark
The above proof contains one moral flaw: all givens have been used exactly once except A0 which was appealed to twice. The reason for this is the packed shape of A1. For the part (A1 ⇐) the restriction to Int.y is necessary, whereas for (A1 ⇒) it is superfluous as far as defining [] is concerned.
(End of Final technical remark.)

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This note has not been written for its mathematical content, which is nill, but for its method of proving which, although standard in my own circles, is fun. More and more computing scientists are getting convinced that the doing of mathematics can be considerably upgraded in its preciseness, vigorousness, and efficiency by adopting what is called "the calculational method". In order to achieve that sooner or later also the traditional mathematician gets involved, we have to collect evidence, as much as we can. Apart from all the criticism that one can have of the recent book of Gries and Schneider, they deserve honour for having given the first shot towards achieving that goal.

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